

L^p Estimates for Semi-Degenerate Simplex Multipliers

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Abstract

C. Muscalu, T. Tao, and C. Thiele prove L^p estimates for a non-degenerate trilinear simplex multiplier called the Biest, which is defined for $(f_1, f_2, f_3) \in \mathcal{S}^3(\mathbb{R})$ by the map

$$C^{1,1,1} : (f_1, f_2, f_3) \mapsto \int_{\xi_1 < \xi_2 < \xi_3} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3.$$

Their methods automatically produce bounds for the collection of all non-degenerate trilinear simplex symbols. Our aim in this article is to prove L^p estimates for a pair of so-called semi-degenerate simplex multipliers given by

$$\begin{aligned} C^{1,1,-2} : (f_1, f_2, f_3) &\mapsto \int_{\xi_1 < \xi_2 < \xi_3} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 - 2\xi_3)} d\xi_1 d\xi_2 d\xi_3 \\ C^{1,1,1,-2} : (f_1, f_2, f_3, f_4) &\mapsto \int_{\xi_1 < \xi_2 < \xi_3 < \xi_4} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) \hat{f}_4(\xi_4) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3 - 2\xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \end{aligned}$$

for which the non-degeneracy condition fails. We obtain as corollaries that $C^{1,1,-2}$ maps into $L^p(\mathbb{R})$ for all $1/2 < p < \infty$ and $C^{1,1,1,-2}$ maps into $L^p(\mathbb{R})$ for all $1/3 < p < \infty$. Both target L^p ranges are shown to be sharp.

1 Introduction

Several recent articles have treated singular integral operators associated to simplexes from a time-frequency perspective. See, for example, [5, 8, 10, 11, 12]. Such objects arise naturally in the asymptotic expansions of solutions to AKNS systems, where estimates of the form $\prod_{i=1}^n L^{p_i'}(\mathbb{R}) \rightarrow L^{\frac{1}{\sum_{i=1}^n \frac{1}{p_i}}}(\mathbb{R})$ are sought for

$$C_n : (f_1, \dots, f_n) \mapsto \sup_t \left| \int_{\xi_1 < \dots < \xi_n < t} \left[\prod_{j=1}^n f_j(\xi_j) e^{2\pi i x(-1)^j \xi_j} \right] d\vec{\xi} \right|.$$

For details on the connection between the family of multisublinear operators $\{C_n\}_{n \geq 1}$ and AKNS, see [1]. It has also been of interest to study the closely related family of Fourier multipliers given for any $\vec{\epsilon} \in \mathbb{R}^n$ and $\vec{f} \in \mathcal{S}^n(\mathbb{R})$ by the formula

$$C^{\vec{\epsilon}} : (f_1, \dots, f_n) \mapsto \int_{\xi_1 < \dots < \xi_n} \hat{f}_1(\xi_1) \dots \hat{f}_n(\xi_n) e^{2\pi i x \vec{\epsilon} \cdot \vec{\xi}} d\vec{\xi}.$$

One highly non-trivial example from the simplex multiplier family is the Biest operator $C^{1,1,1}$, which can be shown to satisfy a wide range of L^p estimates by means of a robust time-frequency argument. More precisely, C. Muscalu, T. Tao, and C. Thiele have the following statement in [12]:

Theorem 1. $C^{1,1,1} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{p_4'}(\mathbb{R})$ as long as $(1/p_1, 1/p_2, 1/p_3, 1/p_4) \in \mathbb{D} \cap \mathbb{D}'$, $1 < p_1, p_2, p_3 \leq \infty$ and $0 < p_4' < \infty$, where \mathbb{D} is the interior of the convex hull of the twelve points

$$\begin{aligned}
D_1 &= \left(1, \frac{1}{2}, 1, -\frac{3}{2}\right) & D_2 &= \left(\frac{1}{2}, 1, 1, -\frac{3}{2}\right) & D_3 &= \left(\frac{1}{2}, 1, -\frac{3}{2}, 1\right) & D_4 &= \left(1, \frac{1}{2}, -\frac{3}{2}, 1\right) \\
D_5 &= \left(1, -\frac{1}{2}, 0, \frac{1}{2}\right) & D_6 &= \left(1, -\frac{1}{2}, \frac{1}{2}, 0\right) & D_7 &= \left(\frac{1}{2}, -\frac{1}{2}, 0, 1\right) & D_8 &= \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \\
D_9 &= \left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right) & D_{10} &= \left(-\frac{1}{2}, 1, \frac{1}{2}, 0\right) & D_{11} &= \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) & D_{12} &= \left(-\frac{1}{2}, \frac{1}{2}, 0, 1\right)
\end{aligned}$$

and \mathbb{D}' is the interior of the convex hull of the collection (D'_1, \dots, D'_{12}) where each D'_j is gotten from the corresponding D_j by swapping the 1st and 3rd positions. For instance, $D'_2 = (1, 1, \frac{1}{2}, -\frac{3}{2})$.

One feature of these estimates is dual index asymmetry. Indeed, for the dual index in positions 3 or 4, we may map near $L^{2/5}(\mathbb{R})$, while in positions 1 and 2 we only map near $L^{2/3}(\mathbb{R})$. Proving the $C^{1,1,1}$ estimates involves splitting $1_{\{\xi_1 < \xi_2 < \xi_3\}} = \tilde{1}_{\mathcal{R}_1} + \tilde{1}_{\mathcal{R}_2} + \tilde{1}_{\mathcal{R}_3}$ into a sum of three symbols localized to the regions $\mathcal{R}_1 = \{|\xi_1 - \xi_2| \gg |\xi_2 - \xi_3|\}$, $\mathcal{R}_2 = \{|\xi_1 - \xi_2| \simeq |\xi_2 - \xi_3|\}$, $\mathcal{R}_3 = \{|\xi_1 - \xi_2| \ll |\xi_2 - \xi_3|\}$ respectively. More precisely, we take $\tilde{1}_{\mathcal{R}_1}$ to be supported inside a set of the form $\{\xi_1 < \xi_2 < \xi_3 : |\xi_1 - \xi_2| \geq C_1 |\xi_2 - \xi_3|\}$ and identically equal to 1 on a set of the form $\{\xi_1 < \xi_2 < \xi_3 : |\xi_1 - \xi_2| \geq C_2 |\xi_2 - \xi_3|\}$ for some constants $C_1 \ll C_2$. A similar statement then holds for both $\tilde{1}_{\mathcal{R}_2}$ and $\tilde{1}_{\mathcal{R}_3}$. As a wide range of L^p estimates hold for the multiplier with symbol $\tilde{1}_{\mathcal{R}_2}$, we focus on estimating the multipliers with symbols $\tilde{1}_{\mathcal{R}_1}$ and $\tilde{1}_{\mathcal{R}_3}$. By symmetry, it suffices to handle $\tilde{1}_{\mathcal{R}_1}$. It is (by now) standard to observe that $T_{\tilde{1}_{\mathcal{R}_1}}$ can be dualized and discretized in time and frequency to yield an average of models sums of type Λ_1 , as clarified by

Definition 1. A model of type Λ_1 is any 4-form writable as

$$\sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1, \Phi_{P_1,1} \rangle \langle f_4, \Phi_{P_2,4} \rangle \left\langle \sum_{\vec{Q} \in \mathbb{Q}: |\omega_Q| < |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2} \rangle \langle f_3, \Phi_{Q_2,3} \rangle \Phi_{Q_3,5}, \Phi_{P_2,0} \right\rangle,$$

where \mathbb{P} is a rank-1 collection of tri-tiles for which $(\omega_{P_1}, \omega_{P_2})$ is adapted to $\{-3\xi_1 = \xi_4\}$, and \mathbb{Q} is a rank-1 collection of tri-tiles for which $(\omega_{Q_1}, \omega_{Q_2})$ is adapted to $\{\xi_1 = \xi_2\}$.

Generalized restricted type estimates for the Biest model Λ_1 are obtained in [12] and then the Marcinkiewicz interpolation yields the desired L^p estimates. For future use, we make the following official definitions:

Definition 2. Let $m : \mathbb{R}^n \rightarrow \mathbb{C}$. Then define the multilinear multiplier T_m on $(f_1, \dots, f_n) \in \mathcal{S}(\mathbb{R})^n$ to be

$$T_m : (f_1, \dots, f_n) \mapsto \int_{\mathbb{R}^n} m(\vec{\xi}) \prod_{j=1}^n \left[\hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi}.$$

Definition 3. For every $\vec{\epsilon} \in \mathbb{R}^n$, let $C^{\vec{\epsilon}}$ denote the n -linear operator defined for all $(f_1, \dots, f_n) \in \mathcal{S}(\mathbb{R})^n$ by the formula

$$C^{\vec{\epsilon}}(f_1, \dots, f_n)(x) = \int_{\xi_1 < \dots < \xi_n} \left[\prod_{j=1}^n \hat{f}_j(\xi_j) e^{2\pi i x \epsilon_j \xi_j} \right] d\vec{\xi}.$$

Definition 4. For every $\vec{\epsilon} \in \mathbb{R}^n$ with only non-zero entries, let $\tilde{C}^{\vec{\epsilon}}$ denote the n -linear operator defined for all $(f_1, \dots, f_n) \in \mathcal{S}(\mathbb{R})^n$ by the formula

$$\tilde{C}^{\vec{\epsilon}}(f_1, \dots, f_n)(x) = \int_{\frac{\epsilon_1}{\epsilon_n} < \dots < \frac{\epsilon_n}{\epsilon_n}} \left[\prod_{j=1}^n \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi}.$$

By construction, for every $\vec{\epsilon} \in \mathbb{R}^n$ with non-zero entries, $\tilde{C}^{\vec{\epsilon}} = T_{1_{\frac{\epsilon_1}{\epsilon_n} < \dots < \frac{\epsilon_n}{\epsilon_n}}}$ and, by a simple change of variables,

$$C^{\vec{\epsilon}}(f_1, \dots, f_n)(x) = \tilde{C}^{\vec{\epsilon}}(f_1(\epsilon_1 \cdot), \dots, f_n(\epsilon_n \cdot))(x) \quad \forall x \in \mathbb{R} \quad \forall (f_1, \dots, f_n) \in \mathcal{S}(\mathbb{R})^n$$

so that $C^{\vec{\epsilon}}$ and $\tilde{C}^{\vec{\epsilon}}$ satisfy the same L^p estimates. We now introduce the following set of definitions:

Definition 5. Let $\vec{\epsilon} \in \mathbb{R}^n$ satisfy the property that there exists a pair (i, j) such that $1 \leq i \leq j \leq n, j-i \in \{0, 1\}$, and $\sum_{k=i}^j \epsilon_k = 0$. Then $\vec{\epsilon}$ is a degenerate tuple and $C^{\vec{\epsilon}}$ is a degenerate simplex multiplier.

Definition 6. Let $\vec{\epsilon} \in \mathbb{R}^n$ satisfy the property that there exists no pair $1 \leq i \leq j \leq n$ such that $\sum_{k=i}^j \epsilon_k = 0$. Then $\vec{\epsilon}$ is a fully non-degenerate tuple and $C^{\vec{\epsilon}}$ is a fully non-degenerate simplex multiplier.

Definition 7. Let $\vec{\epsilon} \in \mathbb{R}^n$ be a non-degenerate tuple for which there exists a pair (i, j) such that $1 \leq i < j \leq n$ and $\sum_{k=i}^j \epsilon_k = 0$. Then $\vec{\epsilon}$ is a semi-degenerate tuple and $C^{\vec{\epsilon}}$ is a semi-degenerate simplex multiplier.

It is important to realize that for any fully non-degenerate $\vec{\epsilon} \in \mathbb{R}^3$ one can use the same argument as before to produce a variant of the Biest model, which still yields the same restricted type estimates as Λ_1 . This is ultimately because the main ingredient needed for proving restricted weak-type estimates is geometric: namely, both tri-tile collections \mathbb{P} and \mathbb{Q} should be adapted to non-degenerate lines in \mathbb{R}^2 , i.e. $l \notin \{\{\xi_1 = 0\}, \{\xi_2 = 0\}, \{\xi_1 + \xi_2 = 0\}\}$. The claim follows by noting that the localized regions of $C^{\epsilon_1, \epsilon_2, \epsilon_3}$ give rise to \mathbb{P} and \mathbb{Q} adapted to non-degenerate lines precisely when $\vec{\epsilon}$ is itself fully non-degenerate. In fact, we have from [13] the following

Theorem 2. Fix $n \geq 1$ and let $\vec{\epsilon} \in \mathbb{R}^n$ be fully non-degenerate. Then $C^{\vec{\epsilon}}$ satisfies a wide range of L^p estimates.

It is not hard to observe that $C^{1,1,-2}$ cannot give rise to a model of type Λ_1 . Therefore, it is natural to ask whether L^p estimates hold in the semi-degenerate setting. As an initial foray, let us discuss one attractive feature of such simplex symbols: they can be broken into simpler pieces, as illustrated by

$$\begin{aligned} \{\xi_1 < \xi_2 < -\xi_3/2\} &= \{\xi_1 + \xi_2 < 2\xi_2 < -\xi_3\} \\ &= \{\xi_1 < \xi_2\} \cap \left[(\{-\xi_3 < \xi_1 + \xi_2\} \cap \{\xi_1 < \xi_2\}) \cup \{\xi_1 + \xi_2 \leq -\xi_3 \leq 2\xi_2\} \right]^c \\ &= \{\xi_1 < \xi_2\} \cap \left[(\{\xi_1 + \xi_2 + \xi_3 > 0\} \cap \{\xi_1 < \xi_2\}) \cup (\{\xi_1 + \xi_2 + \xi_3 \leq 0\} \cap \{-\xi_3 \leq 2\xi_2\}) \right]^c. \end{aligned}$$

This elegant observation is due to C. Muscalu. Using $H^+ = T_{\{\xi > 0\}}$ and $H^- = T_{\{\xi \leq 0\}}$, the above decomposition yields the identity

$$\tilde{C}^{1,1,-1/2}(f_1, f_2, f_3)(x) = \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - H^+(\tilde{C}^{1,1}(f_1, f_2) \cdot f_3)(x) - H^-(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x).$$

Because each term on the RHS of the above display satisfies all interior Banach estimates, the same must be true for $\tilde{C}^{1,1,-1/2}$ and therefore $C^{1,1,-2}$. Given that $C^{1,1,-2}$ maps into $L^r(\mathbb{R})$ for all $1 < r < \infty$, it is tempting to ask whether such an object can map below $L^1(\mathbb{R})$, and if so, how low can the target exponent $r \geq \frac{1}{3}$ go. Our first result shows $r > 1/2$ is necessary for $C^{1,1,-2}$ to map into $L^r(\mathbb{R})$. Similarly, we have the identity

$$\begin{aligned} &\tilde{C}^{1,1,1,-1/2}(f_1, f_2, f_3, f_4)(x) \\ &= \tilde{C}^{1,1,1}(f_1, f_2, f_3)(x) f_4(x) - T_{\{\xi_1 < \xi_2 < \xi_3\} \cap \{\xi_2 + \xi_3 + \xi_4 > 0\}}(f_1, f_2, f_3, f_4)(x) - \tilde{C}^{1,1,-1}(f_1, f_2, \tilde{C}^{-1/2,1}(f_4, f_3))(x). \end{aligned}$$

Because both $T_{\{\xi_1 < \xi_2 < \xi_3\} \cap \{\xi_2 + \xi_3 + \xi_4 > 0\}}$ and $\tilde{C}^{1,1,-1}(f_1, f_2, \tilde{C}^{-1/2,1}(f_4, f_3))$ satisfy no L^p estimates, so any bounds for $\tilde{C}^{1,1,1,-1/2}$ must arise as a consequence of large destructive interference between these two unbounded terms. A natural question in light of these developments is whether the degeneracy condition is necessary for L^p estimates to fail. While not answering this question fully, we content ourselves in this section with establishing two principle results. The first is

Theorem 3. $C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{p'_4}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p'_4$, and $(p_1, p_2, p_3, p_4) \in \text{Int}(\text{Conv}[\mathcal{A}]) \cap \text{Int}(\text{Conv}[\mathcal{A}'])$, where $\mathcal{A} = \{A_j\}_{j=1}^9$ is given by

$$\begin{aligned} A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right) \\ A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \\ A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right) \end{aligned}$$

and \mathcal{A}' denotes the collection $\{A'_1, \dots, A'_9\}$ where each A'_j is gotten by the corresponding A_j by swapping the 1st and 3rd indices. For example, $A'_2 = (1, \frac{1}{2}, \frac{1}{2}, -1)$. For any set of points $S \subset \mathbb{R}^n$, we use $\text{Int}(\text{Conv}([S]))$ to denote the interior of the convex hull of the set S .

To prove Theorem 5, we follow the standard procedure introduced in [12] of carving $1_{\xi_1 < \xi_2 < -\xi_3/2}$ into three localized pieces, discretizing each piece into a wave packet model, and then obtaining satisfactory estimates for each model. Central to our argument will be producing generalized restricted type estimates for models of type Λ_2 , as clarified by

Definition 8. A model of type Λ_2 is any 4-form writable as

$$\sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1, \Phi_{P_{1,1}} \rangle \langle f_4, \Phi_{P_{2,4}} \rangle \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}: \omega_{Q_3} \subset \subset \omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_3, \Phi_{Q_{1,3}}^\alpha \rangle \langle f_4, \Phi_{Q_{2,4}}^\alpha \rangle \Phi_{Q_{3,5}}^\alpha d\alpha, \Phi_{P_{2,0}} \right\rangle,$$

where \mathbb{P} is a collection of tri-tiles for which $(\omega_{P_1}, \omega_{P_3})$ is adapted to $\{\xi_1 + \xi_2 = 0\}$, $\Phi_{P_{2,4}}$ is lacunary about the origin at scale $|\omega_P|$, and \mathbb{Q} is a rank-1 collection of tri-tiles for which $(\omega_{Q_1}, \omega_{Q_2})$ is adapted to $\{\xi_1 = \xi_2\}$ and $\Phi_{Q_{1,3}}^\alpha$ is a wave packet on Q_1 uniformly for $\alpha \in [0, 1]$. The implicit constant in the \mathbb{Q} -sum should be taken to be some sufficiently large absolute constant. Moreover, $\omega_{Q_2} \subset \subset \omega_{P_2}$ means $|\omega_{Q_2}| \ll |\omega_{P_2}|$ and $\omega_{Q_2} \subset \omega_{P_2}$.

Our second principle result is

Theorem 4. $C^{1,1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \times L^{p_4}(\mathbb{R}) \rightarrow L^{p'_4}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p'_4 < \infty$ and $(p_1, p_2, p_3, p_4) \in \text{Int}(\text{Conv}[\mathcal{B}]) \cap \text{Int}(\text{Conv}[\mathcal{B}'])$, where $\mathcal{B} = \{B_j\}_{j=1}^{16}$ is given by

$$\begin{aligned} B_1 &= \left(1, 1, \frac{1}{2}, \frac{1}{2}, -2\right), B_2 = \left(1, \frac{1}{2}, \frac{1}{2}, 1, -2\right), B_3 = \left(1, \frac{1}{2}, 1, \frac{1}{2}, -2\right) \\ B_4 &= \left(-2, 1, \frac{1}{2}, \frac{1}{2}, 1\right), B_5 = \left(-2, \frac{1}{2}, 1, \frac{1}{2}, 1\right), B_6 = \left(-2, \frac{1}{2}, \frac{1}{2}, 1, 1\right) \\ B_7 &= \left(0, -\frac{3}{2}, \frac{1}{2}, 1, 1\right), B_8 = \left(1, -\frac{3}{2}, \frac{1}{2}, 1, 0\right), B_9 = \left(0, -\frac{3}{2}, 1, \frac{1}{2}, 1\right), B_{10} = \left(1, -\frac{3}{2}, 1, \frac{1}{2}, 0\right) \\ B_{11} &= \left(0, \frac{1}{2}, -\frac{1}{2}, 1, 0\right), B_{12} = \left(\frac{1}{2}, 0, -\frac{1}{2}, 1, 0\right), B_{13} = \left(0, 0, -\frac{1}{2}, 1, \frac{1}{2}\right) \\ B_{14} &= \left(0, \frac{1}{2}, 1, -\frac{1}{2}, 0\right), B_{15} = \left(\frac{1}{2}, 0, 1, -\frac{1}{2}, 0\right), B_{16} = \left(0, 0, 1, -\frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

and \mathcal{B}' denotes the collection $\{B'_j\}_{j=1}^{16}$, where each B'_j is obtained from the corresponding B_j by the permutation $1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 2, 4 \mapsto 3$. In particular, $B'_3 = (1, 1, \frac{1}{2}, \frac{1}{2}, -2)$. Moreover, $(1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3} - 2) \in \overline{\text{Conv}[\mathcal{B}]} \cap \overline{\text{Conv}[\mathcal{B}]}$ and $C^{1,1,1,-2}$ maps into $L^r(\mathbb{R})$ for all $\frac{1}{3} < r \leq 1$.

Unlike $C^{1,1,-2}$, we do not have a natural decomposition of $C^{1,1,1,-2}$ into simpler bounded operators. Hence, it is perhaps a little surprising that $C^{1,1,1,-2}$ satisfies any L^p estimates. At the end of the day, we are able to reduce matters to proving generalized restricted type estimates for models of Λ_3 type, as clarified by

Definition 9. A model of type Λ_3 is any 5-form writable as

$$\Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}(f_1, f_2, f_3, f_4, f_5) = \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R} \in \mathbb{R}: \omega_{R_1} \supset \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_{1,1}} \rangle \langle f_5, \Phi_{R_{2,5}} \rangle}{|I_{\vec{R}}|^{1/2}} \Phi_{R_{3,0}}^{n-l}, \Phi_{P_{1,6}} \right\rangle \langle f_2, \Phi_{P_{2,2}} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_{3,7}} \right\rangle,$$

where \mathbb{P}, \mathbb{Q} , and \mathbb{R} are three tri-tile collections, \mathbb{Q} is rank-1, and for each $\alpha \in [0, 1]$,

$$BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4)(x) := \sum_{\vec{Q} \in \mathbb{Q}: \omega_{Q_3} \subset \subset \omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_3, \Phi_{Q_{1,3}}^\alpha \rangle \langle f_4, \Phi_{Q_{2,4}}^\alpha \rangle \Phi_{Q_{3,5}}^\alpha(x)$$

where each wave packet $\Phi_{Q_{i,j(i)}}^\alpha$ for $i = 1, 2, 3$ has decay constant uniform in α .

An attractive feature of our main results is that the L^p target ranges for both $C^{1,1,-2}$ and $C^{1,1,1,-2}$ are the best possible. Indeed, that $C^{1,1,-2}$ cannot map below $L^{\frac{1}{2}}(\mathbb{R})$ and $C^{1,1,1,-2}$ cannot map below $L^{\frac{1}{3}}(\mathbb{R})$ follows from explicit counterexamples in §2. This sharpness is quite different from the fully non-degenerate setting, where the generic *BHT* model produces estimates only down to $L^{\frac{2}{3}+\epsilon}(\mathbb{R})$ and there are no known counterexamples at this time to rule out the *BHT* mapping all the way down to $L^{\frac{1}{2}+\epsilon}(\mathbb{R})$.

Theorems 5 and 8 also prompt another line of questioning; do we have the same L^p estimates if the corresponding symbols $1_{\{\xi_1 < \xi_2 < -\xi_3/2\}}$ and $1_{\{\xi_1 < \xi_2 < \xi_3 < -\xi_4/2\}}$ are respectively replaced with $m_1(\xi_1, \xi_2, \xi_3) = b_1(\xi_1, \xi_2)b_2(\xi_2, \xi_3)$ and $m_2(\xi_1, \xi_2, \xi_3, \xi_4) = c_1(\xi_1, \xi_2)c_2(\xi_3, \xi_3)c_3(\xi_3, \xi_4)$, where b_1, c_1, c_2 are adapted to $\{\xi_1 = \xi_2\}$, and b_2, c_3 are adapted to $\{\xi_1 = -\xi_2/2\}$? The answer is assuredly yes; however, the proofs in the generic case become longer, less reader-friendly, and tend to obscure the important points of the semi-degenerate analysis, and so the details of the argument are omitted. Nonetheless, we have all the tools necessary to carry out the proof and now provide the briefest possible sketch. Generic trilinear multipliers $m(\xi_1, \xi_2, \xi_3)$ of the form $b_1(\xi_1, \xi_2) \cdot b_2(\xi_2, \xi_3)$ may be reduced to models of type Λ_2 combined with error terms with even better mapping properties by following the arguments in [3]. Showing the same estimates for generic 4-linear multipliers of the form $c_1(\xi_1, \xi_2)c_2(\xi_2, \xi_3)c_3(\xi_3, \xi_4)$ requires us to mimic our local discretization of the form associated to a regional piece of

$$B[a_1, a_2] : (f_1, f_2, f_3) \mapsto \int_{\mathbb{R}} a_1(\xi_1, \xi_2) a_2(\xi_2, \xi_3) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3$$

for any $a_1, a_2 : \mathbb{R}^2 \rightarrow \mathbb{C}$ adapted to the degenerate line $\{\xi_1 + \xi_2 = 0\}$ and then deploy the l^1 energy boost from the proof of Theorem 5. Before describing the counterexamples and positive results in the semi-degenerate setting, we should say a bit about the bigger picture. What can be said for $C^{1,1,1,1,-2}$ or, for that matter, any $C^{\vec{\epsilon}}$ with $\vec{\epsilon} \in \mathbb{R}^n$ semi-degenerate? One expects that if any such simplex multiplier satisfied no L^p estimates, then $C^{1,1,1,-2}$ should fail to have L^p estimates. Indeed, as we see two bad best lurking in our natural decomposition of $C^{1,1,1,-2}$, it is reasonable to expect that matters cannot really deteriorate beyond such these dueling bad bests. As Theorem 8 ensures a wide range of estimates for $C^{1,1,1,-2}$, we are naturally led to

Conjecture 1. *Let $\vec{\epsilon} \in \mathbb{R}^n$ be semi-degenerate. Then $C^{\vec{\epsilon}}$ satisfies a wide range of L^p estimates.*

Given Theorem 2 and the existence of generic mixed estimates for degenerate simplex multipliers, the resolution of Conjecture 1 would in some sense complete the picture of simplex multiplier estimates.

2 Terry Lyons' Variational Estimate

Closely related to the *a.e.* converge of the Fourier series of L^p functions are the fundamental estimates of Carleson and Hunt, which asserts that the map initially defined for $f \in \mathcal{S}(\mathbb{R})$ by the rule

$$C : f \mapsto \sup_{N \in \mathbb{R}} \left| \int_{(\infty, N]} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

can be extended to all of $L^p(\mathbb{R})$ and satisfies $\|C(f)\|_p \lesssim_p \|f\|_p$ for all $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. The variational Carleson estimates are a generalization of this result: for any $2 < \rho \leq \infty$,

$$\mathcal{C}^\rho : f \mapsto \sup_{k \in \mathbb{N}} \sup_{\xi_1 < \xi_2 < \dots < \xi_k} \left(\sum_{n=1}^{k-1} \left| \int_{\xi_n < \eta < \xi_{n+1}} \hat{f}(\eta) e^{2\pi i x \eta} d\eta \right|^\rho \right)^{1/\rho}$$

extends to a map of $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ for all $\rho' < p < \infty$. When $\rho = \infty$, we clearly recover the Carleson estimates. It is known via a direct counterexample appearing in the author's previous work that $\rho > 2$ is necessary for any L^p estimates to hold. In light of the variational Carleson story, it is natural to ask whether estimates hold for the variational Bi-Carleson, which is defined for variation exponent $0 < \rho \leq \infty$ and with domain $\mathcal{S}(\mathbb{R})^2$ to be

$$\mathcal{BC}^\rho : (f_1, f_2) \mapsto \sup_{k \in \mathbb{N}} \sup_{\xi_1 < \xi_2 < \dots < \xi_k} \left(\sum_{n=1}^{k-1} \left| \int_{\xi_n < \eta_1 < \eta_2 < \xi_{n+1}} \hat{f}_1(\eta_1) \hat{f}_2(\eta_2) e^{2\pi i x(\eta_1 + \eta_2)} d\eta_1 d\eta_2 \right|^\rho \right)^{1/\rho}.$$

If $\rho = \infty$, then \mathcal{BC}^∞ is the Bi-Carleson operator, for which estimates were obtained in [8] and shown to coincide with the known *BHT* estimates $1 < p_1, p_2 \leq \infty$ and $0 < \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$. Moreover, one expects $\mathcal{BC}^{1+\epsilon} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ for every $\epsilon > 0$, provided the *BHT* behaves like a product. Indeed, in this simple case, we could deduce the desired $\rho = 1+\epsilon$ estimate by concatenating Cauchy-Schwarz with the variational Carleson estimate near $\rho = 2$. Interpolating between the $\rho = 1+\epsilon$ and $\rho = \infty$ cases would then yield

$$\mathcal{BC}^\rho : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^{\frac{p_1 p_2}{p_1 + p_2}}(\mathbb{R})$$

for all $1 < p_1, p_2 < \infty$ such that $0 < \frac{1}{p_1} + \frac{1}{p_2} < 1 + \frac{1}{2\rho'}$, $\max\{\frac{1}{p_1}, \frac{1}{p_2}\} < \frac{1}{2} + \frac{1}{2\rho'}$. We next present a striking inequality due to Terry Lyons in [7], which provides a pointwise bound for trilinear simplex multipliers in terms of various powers of the variational Carleson and Bi-Carleson operators. For all $2 < r < 3$, we in fact have

$$\begin{aligned} C^{1,1,1}(f_1, f_2, f_3)(x) &\leq [\text{Var}^r(f_1, f_2, f_3)(x)]^3 \\ \text{Var}^r(f_1, f_2, f_3)(x) &:= \mathcal{C}^r(f_1)(x) + \mathcal{C}^r(f_2)(x) + \mathcal{C}^r(f_3)(x) \\ &\quad + \left[\mathcal{BC}^{r/2}(f_1, f_2)(x)\right]^{1/2} + \left[\mathcal{BC}^{r/2}(f_2, f_3)(x)\right]^{1/2} + \left[\mathcal{BC}^{r/2}(f_1, f_3)(x)\right]^{1/2}. \end{aligned}$$

Taking $r \simeq 3$ and $p_1 = p_2 = p_3 \simeq 3/2$ and using the variational Carleson and variational Bi-Carleson estimates gives the extremal mapping $L^{3/(2-\epsilon)} \times L^{3/(2-\epsilon)} \times L^{3/(2-\epsilon)} \rightarrow L^{1/(2-\epsilon)}$. By interpolation, one recovers all estimates in the convex hull of $\mathcal{S} := \mathcal{B} \cup (2/3 - \epsilon/3, 2/3 - \epsilon/3, 2/3 - \epsilon/3, -1 + \epsilon)$, where \mathcal{B} denotes the set of all interior Banach estimates, i.e. $\mathcal{B} = \left\{(1/p_1, 1/p_2, 1/p_3, 1 - 1/p_1 - 1/p_2 - 1/p_3) : 1 < p_1, p_2, p_3 < \infty, \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1\right\}$. Our proof of Theorem 5 has the two-fold advantage of avoiding reliance on the variational Bi-Carleson/Carleson estimates and producing estimates beyond the convex hull of \mathcal{S} .

3 $C^{1,1,-2}$ and $C^{1,1,1,-2}$ Counterexamples

We begin with

Proposition 1. $C^{1,1,-2}$ does not map into $L^r(\mathbb{R})$ for $r \leq 1/2$.

Proof. Fix $1 < p_1, p_2, p_3 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 2$. Let $f_1 = f_2 = f_3 = \check{1}_{[-1,1]}$. Then $\prod_{j=1}^3 \|f_j\|_{p_j} < \infty$. Note

$$\begin{aligned} C^{1,1,-2}(f_1, f_2, f_3)(x) &= \int_{-1 < \xi_1 < \xi_2 < \xi_3 < 1} e^{2\pi i x (\xi_1 + \xi_2 - 2\xi_3)} d\xi_1 d\xi_2 d\xi_3 \\ &= \frac{1}{2\pi i x} \int_{-1 < \xi_2 < \xi_3 < 1} \left[e^{2\pi i x (2\xi_2 - 2\xi_3)} - e^{2\pi i x (-1 + \xi_2 - 2\xi_3)} \right] d\xi_2 d\xi_3 \\ &= \left[\frac{1}{2\pi i x} \right]^2 \int_{-1 < \xi_3 < 1} \left[1/2 - e^{2\pi i x (-1 - \xi_3)} - e^{2\pi i x (-2 - 2\xi_3)} / 2 + e^{2\pi i x (-2 - 2\xi_3)} \right] d\xi_3 \\ &= \left[\frac{1}{2\pi i x} \right]^2 \int_{-1 < \xi_3 < 1} \left[1/2 - e^{2\pi i x (-1 - \xi_3)} - e^{2\pi i x (-2 - 2\xi_3)} / 2 \right] d\xi_3 \\ &= \left[\frac{1}{2\pi i x} \right]^2 + \left[\frac{1}{2\pi i x} \right]^3 \left[5/4 - e^{-4\pi i x} - e^{-8\pi i x} / 4 \right]. \end{aligned}$$

Therefore, $C^{1,1,-2}(\vec{f})(x)$ decays like $\frac{1}{|x|^2}$ away from the origin ($|x| \gtrsim 1$) and so cannot belong to $L^r(\mathbb{R})$ for $r \leq 1/2$. If $p_i = 1$ for some $j \in \{1, 2, 3\}$, then one can instead take $f_1 = f_2 = f_3 = \mathcal{F}^{-1}[\phi]$ for some non-trivial, non-negative $\phi \in C_{[-1,1]}^\infty(\mathbb{R})$ and use integration by parts to deduce the same quadratic decay as before. \square

The analogous statement for $C^{1,1,1,-2}$ is

Proposition 2. $C^{1,1,1,-2}$ does not map into $L^r(\mathbb{R})$ for $r \leq \frac{1}{3}$.

Proof. If $C^{1,1,1,-2}$ did map into $L^r(\mathbb{R})$ for some $r \leq \frac{1}{3}$, there would exist a 4-tuple (p_1, p_2, p_3, p_4) satisfying $1 \leq p_1, p_2, p_3, p_4 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} \geq 3$ for which

$$\|C^{1,1,1,-2}(f_1, f_2, f_3, f_4)\|_{L^{\frac{1}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}}}(\mathbb{R})} \lesssim_{\vec{p}} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})} \|f_3\|_{L^{p_3}(\mathbb{R})} \|f_4\|_{L^{p_4}(\mathbb{R})}$$

for all $f_j \in L^{p_j}(\mathbb{R})$ and $j \in \{1, 2, 3, 4\}$.

CASE #1: $\frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} > 2$. Then take $f_2 = f_3 = f_4 = \mathcal{F}^{-1}[\phi]$ along with $f_1^N = \mathcal{F}^{-1}[Dil_{N^{-1}}^1 Tr_{-2N} \phi] = \mathcal{F}^{-1}[\phi](N^{-1}x)e^{-2\pi i 2x}$ where ϕ is again some non-trivial, non-negative function in $C_{[-1,1]}^\infty(\mathbb{R})$. Then for large enough N , $C^{1,1,-2}(f_1^N, f_2, f_3, f_4) = f_1^N(x)C^{1,1,-2}(f_2, f_3, f_4)(x)$, and so

$$\|C^{1,1,-2}(f_1^N, f_2, f_3, f_4)\|_{L^{\frac{1}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}}}(\mathbb{R})} \simeq N^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} - 2},$$

whereas $\|f_1^N\|_{L^{p_1}(\mathbb{R})} \prod_{j=2}^4 \|f_j\|_{L^{p_j}(\mathbb{R})} \simeq N^{1/p_1}$. Taking N arbitrarily large contradicts our original assumption.

CASE #2: $\frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 2$. Then $p_1 = 1$. Setting $f_1^N(x) = \mathcal{F}^{-1}[Dil_{N^{-1}}^1 Tr_{-2N} \phi](x)\mathcal{F}^{-1}[1_{[-1,1]}](x)$ for the same ϕ as before ensures that $C^{1,1,1,-2}(f_1^N, f_2, f_3, f_4)(x) = f_1^N(x)C^{1,1,-2}(f_2, f_3, f_4)(x)$ for large enough N . Hence,

$$\|C^{1,1,1,-2}(f_1^N, f_2, f_3, f_4)\|_{L^{\frac{1}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}}}(\mathbb{R})} \simeq (\ln N)^3,$$

whereas $\|f_1^N\|_{L^{p_1}(\mathbb{R})} \prod_{j=2}^4 \|f_j\|_{L^{p_j}(\mathbb{R})} \simeq \ln N$. Taking N arbitrarily large again contradicts our original assumption. \square

4 $C^{1,1,-2}$ Estimates

Our goal in this section is to prove

Theorem 5. $C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{p'_4}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p'_4$, and $(p_1, p_2, p_3, p_4) \in \text{Int}(\text{Conv}[\mathcal{A}]) \cap \text{Int}(\text{Conv}[\mathcal{A}'])$, where $\mathcal{A} = \{A_j\}_{j=1}^9$ is given by

$$\begin{aligned} A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right) \\ A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \\ A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right) \end{aligned}$$

and \mathcal{A}' denotes the collection $\{A'_1, \dots, A'_9\}$ where each A'_j is gotten by the corresponding A_j by swapping the 1st and 3rd indices. For example, $A'_2 = (1, \frac{1}{2}, \frac{1}{2}, -1)$.

It is clear that \mathcal{C} strictly contains the estimates obtained by interpolating between the diagonal

$$\Delta := \{(p, p, p, 1 - 3p) : 1/3 \leq p < 2/3\}$$

and the interior Banach estimates $\mathbb{B} = \{\vec{p} : 1 < p_j \leq \infty \forall j \in \{1, 2, 3, 4\}\}$, so our range is providing estimates not obtainable from Terry Lyon's estimate and the variational Bi-Carleson estimates. For instance, $C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{p'_4}(\mathbb{R})$ for tuples (p_1, p_2, p_3, p_4) in a small neighborhood of $(1, \frac{1}{2}, \frac{1}{2}, -1)$.

4.1 Reduction to the Model

Our analysis of the simplex multiplier $\tilde{C}^{1,1,-2}$ begins as in the Biest case by localizing the symbol $1_{\xi_1 < \xi_2 < -\xi_3/2}$ inside the three regions:

$$\begin{aligned}\mathcal{R}_1 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| << |\xi_2 + \xi_3/2|\} \\ \mathcal{R}_2 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| \simeq |\xi_2 + \xi_3/2|\} \\ \mathcal{R}_3 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| >> |\xi_2 + \xi_3/2|\}.\end{aligned}$$

To this end, let us recall

$$1_{\{\xi_2 < -\xi_3/2\}}(\xi_2, \xi_3) = \sum_{(\gamma, \gamma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{k, k' \in \mathbb{Z}} \sum_{\vec{Q} \in \mathcal{Q}^{\gamma, \gamma'}} c_k \tilde{c}_{k'} \hat{\eta}_{\omega_{Q_1}, 2}^{\gamma, k}(\xi_2) \hat{\eta}_{\omega_{Q_2}, 3}^{\gamma', k'}(\xi_3)$$

where γ, γ' are dyadic shifts, k, k' are oscillation parameters, and each $\vec{Q} = (\omega_{Q_2}, \omega_{Q_3})$ is a Whitney square for the set $\Gamma := \{\xi_2 = -\xi_3/2\}$ in the usual sense that the side-length of \vec{Q} is proportional to $\text{dist}(\vec{Q}, \Gamma)$. Similarly, we have

$$1_{\{\xi_1 < \xi_2\}}(\xi_1, \xi_2) = \sum_{(\sigma, \sigma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{l, l' \in \mathbb{Z}} \sum_{\vec{P} \in \mathcal{P}^{\sigma, \sigma'}} c_l \tilde{c}_{l'} \hat{\eta}_{\omega_{P_1}, 1}^{\sigma, l}(\xi_1) \hat{\eta}_{\omega_{P_2}, 0}^{\sigma', l'}(\xi_2)$$

where we have the same setup as before with the exception that each $\vec{P} = (\omega_{P_1}, \omega_{P_2})$ is a Whitney cube for the set $\tilde{\Gamma} := \{\xi_1 = \xi_2\}$. The main trick we want to use is that inside \mathcal{R}_3 , say $\xi_1 < \xi_2 < -\xi_3/2$ holds iff $\xi_1 < -(\xi_2 + \xi_3)$; $\xi_2 < -\frac{\xi_3}{2}$ holds. Therefore, setting

$$\begin{aligned}\tilde{1}_{\mathcal{R}_3}(\xi_1, \xi_2, \xi_3) &= \sum_{(\gamma, \gamma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{k, k' \in \mathbb{Z}} \sum_{\vec{Q} \in \mathcal{Q}^{\gamma, \gamma'}} c_k \tilde{c}_{k'} \hat{\eta}_{\omega_{Q_1}, 2}^{\sigma, l}(\xi_2) \hat{\eta}_{\omega_{Q_2}, 3}^{\sigma', l'}(\xi_3) \\ &\times \left[\sum_{(\sigma, \sigma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{l, l' \in \mathbb{Z}} \sum_{\vec{P} \in \mathcal{P}^{\sigma, \sigma'}: |\vec{P}| >> |\vec{Q}|} c_l \tilde{c}_{l'} \hat{\eta}_{\omega_{P_1}, 1}^{\gamma, k}(\xi_1) \hat{\eta}_{\omega_{P_2}, 0}^{\gamma', k'}(-(\xi_2 + \xi_3)) \right],\end{aligned}$$

it follows that for large enough implicit constant, $\tilde{1}_{\mathcal{R}_3}(\xi_1, \xi_2, \xi_3) \equiv 1$ on a set \mathcal{R}_3^0 and supported on a set of the same shape \mathcal{R}_3^1 . We may similarly construct $\tilde{1}_{\mathcal{R}_1}$. Then putting it all together yields

$$\begin{aligned}&1_{\xi_1 < \xi_2 < -\xi_3/2} \\ &= 1_{\xi_1 < \xi_2 < -\xi_3/2} (1 - \tilde{1}_{\mathcal{R}_1})(1 - \tilde{1}_{\mathcal{R}_2}) + 1_{\xi_1 < \xi_2 < -\xi_3/2} \tilde{1}_{\mathcal{R}_1} + 1_{\xi_1 < \xi_2 < -\xi_3/2} \tilde{1}_{\mathcal{R}_2} - 1_{\xi_1 < \xi_2 < -\xi_3/2} \tilde{1}_{\mathcal{R}_1} \tilde{1}_{\mathcal{R}_2} \\ &:= I + II + III + IV.\end{aligned}$$

It is straightforward to observe that I is Mikhlin-Hörmander symbol adapted to the set of shape \mathcal{R}_2 and $IV \equiv 0$ for large enough implicit constants. Therefore, it suffices to understand II and III . However, by construction,

$$\begin{aligned}1_{\xi_1 < \xi_2 < -\xi_3/2} \tilde{1}_{\mathcal{R}_1} &= \tilde{1}_{\mathcal{R}_1} \\ 1_{\xi_1 < \xi_2 < -\xi_3/2} \tilde{1}_{\mathcal{R}_3} &= \tilde{1}_{\mathcal{R}_3}\end{aligned}$$

so it suffices (morally speaking) to obtain estimate for $\tilde{1}_{\mathcal{R}_1}$ and $\tilde{1}_{\mathcal{R}_3}$. Moreover, by symmetry, it suffices (morally speaking) to obtain estimates for $\tilde{1}_{\mathcal{R}_3}$. Of course, we will need to write down estimates for symbols adapted to $\{\xi_1 = \xi_2 = -\xi_3/2\}$, but this argument will be postponed until later. To handle $\tilde{1}_{\mathcal{R}_3}$, we now wish to dualize by introducing f_4 and complete as follows:

$$\begin{aligned}
& \int_{\mathbb{R}} T_{\bar{I}_{\mathcal{R}_3}}(f_1, f_2, f_3)(x) f_4(x) dx \\
&= \sum' c(k, k', l, l') \int_{\mathbb{R}} f_1 * \eta_{\omega_{P_1}, 1}^{\gamma, k} \cdot \left[f_2 * \eta_{\omega_{Q_1}, 2}^{\sigma, l} f_3 * \eta_{\omega_{Q_2}, 3}^{\sigma', l'} \right] * \eta_{-\omega_{P_2}, 0}^{\gamma', k'} \cdot f_4 dx \\
&= \sum' c(k, k', l, l') \int_{\mathbb{R}} f_1 * \eta_{\omega_{P_1}, 1}^{\gamma, k} \cdot \left[f_2 * \eta_{\omega_{Q_1}, 2}^{\sigma, l} f_3 * \eta_{\omega_{Q_2}, 3}^{\sigma', l'} \right] * \eta_{-\omega_{P_2}, 0}^{\gamma', k'} \cdot f_4 * \eta_{|\vec{P}|, 4}^{lac} dx \\
&= \sum' c(k, k', l, l') \int_{\mathbb{R}} \left(\left[f_1 * \eta_{\omega_{P_1}, 1}^{\gamma, k} f_4 * \eta_{|\vec{P}|, 4}^{lac} \right] * \eta_{\omega_{P_2}, 0}^{\gamma', k'} \right) * \eta_{\omega_{Q_3}, 5} \cdot f_2 * \eta_{\omega_{Q_1}, 2}^{\sigma, l} \cdot f_3 * \eta_{\omega_{Q_2}, 3}^{\sigma', l'} dx,
\end{aligned}$$

where $\omega_{Q_3} \supset -\omega_{Q_1} - \omega_{Q_2}$. We are now pleased with the above integral expression and may proceed to discretize in time with respect to the \mathcal{Q} and \mathcal{P} Whitney cubes. The details required for this process are routine and so are omitted. At the end of the day,

$$\int_{\mathbb{R}} \left(\left[f_1 * \eta_{\omega_{P_1}, 1}^{\gamma, k} f_4 * \eta_{|\vec{P}|, 4}^{lac} \right] * \eta_{\omega_{P_2}, 0}^{\gamma', k'} \right) * \eta_{\omega_{Q_3}, 5} \cdot f_2 * \eta_{\omega_{Q_1}, 2}^{\sigma, l} \cdot f_3 * \eta_{\omega_{Q_2}, 3}^{\sigma', l'} dx$$

can be written as

$$\sum' \int_0^1 \int_0^1 \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1, \Phi_{P_1, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{P_4, 4}^{\alpha'} \rangle \left\langle \sum_{\vec{Q} \in \mathbb{Q}: |\omega_{\vec{Q}}| < |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1, 2}^{\alpha, \sigma, l} \rangle \langle f_3, \Phi_{Q_2, 3}^{\alpha, \sigma', l'} \rangle \Phi_{Q_3, 5}^{\alpha} \Phi_{P_2, 0}^{\alpha', \gamma', k'} \right\rangle d\alpha d\alpha',$$

where \mathbb{Q} is a rank-1 collection of tri-tiles, where \mathbb{P} is a collection of tri-tiles adapted to the degenerate line $\{\xi_1 + \xi_2 = 0\}$. Each tri-tile $\vec{P} = (P_1, P_2, P_3)$, where each $P_j = (I_{\vec{P}}, \omega_{P_j})$ is a tile. Moreover, each $\Phi_{P, j}$ is a wave-packet on the tile P for each $j \in \{0, 1, 2, 3, 4, 5\}$.

Definition 10. Let $n \geq 1$ and $\sigma \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. We define the shifted n -dyadic mesh $D = D_{\sigma}^n$ to be the collection of cubes of the form

$$D_{\sigma}^n := \{2^j(k + (0, 1)^n + (-1)^j \sigma) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

Observe that for every cube Q , there exists a shifted dyadic cube Q' such that $Q \subseteq \frac{7}{10}Q'$ and $|Q'| \sim |Q|$; this property clearly follows from verifying the $n = 1$ case. The constant $\frac{7}{10}$ is not especially important here.

Definition 11. A subset D' of a shifted n -dyadic grid D is called *sparse*, if for any two cubes Q, Q' in D with $Q \neq Q'$ we have $|Q| < |Q'|$ implies $10^9 Q \subset |Q'|$ and $|Q| = |Q'|$ implies $10^9 Q \cap 10^9 Q' = \emptyset$.

It is immediate from the above definition that any subset of a shifted n -dyadic grid can be split into $O(C^n)$ sparse subsets.

Definition 12. For a given spatial interval I , let $\tilde{\chi}_I(x) := \left(1 + \left(\frac{|x - x_I|}{|I|}\right)^2\right)^{1/2}$, where x_I is the center of I .

Definition 13. Let $P = (I_P, \omega_P)$ be a tile. A wave packet on P is a function Φ_P which has Fourier support in $\frac{9}{10}\omega_P$ and obeys the estimate

$$|\Phi_P(x)| \lesssim_M |I_P|^{-1/2} \tilde{\chi}_{I_P}^M(x)$$

for some fixed large integer M . Therefore, Φ_P is L^2 normalized and adapted to the Heisenberg box (I_P, ω_P) .

We next introduce the tile ordering $<$ from [12], which is in the spirit of Fefferman or Lacey and Thiele, but different inasmuch as P' and P do not have to intersect.

Definition 14. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$, and let $1 \leq i \leq 3$. An i -tile with shift σ_i is a rectangle $P = (I_P, \omega_P)$ with area 1 and with $I_P \in D_0^1, \omega_P \in D_{\sigma_i}^1$. A tri-tile with shift σ is a 3-tuple $\vec{P} = (P_1, P_2, P_3)$ such that each P_i is an i -tile with shift σ_i , and the $I_{P_i} = I_{\vec{P}}$ are independent of i . The frequency cube $Q_{\vec{P}}$ of a tri-tile is defined to be $\prod_{i=1}^3 \omega_{P_i}$.

Definition 15. A set \mathbb{P} of tri-tiles is called *sparse*, if all the tri-tiles in \mathbb{P} have the same shift σ and the set of frequency cubes $\{Q_{\vec{P}} = (\omega_{P_1}, \omega_{P_2}, \omega_{P_3}) : \vec{P} \in \mathbb{P}\}$ is sparse.

Definition 16. Let P and P' be tiles. We write $P' < P$ if $I_{P'} \subsetneq I_P$ and $3\omega_P \subseteq 3\omega_{P'}$, and $P' \leq P$ if $P' < P$ or $P' = P$. We write $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $10^7\omega_P \subseteq 10^7\omega_{P'}$. We write $P' \lesssim' P$ if $P' \lesssim P$ and $P' \not\leq P$.

Definition 17. A collection \mathbb{P} of tri-tiles is said to have *rank 1* if one has the following properties for all $\vec{P}, \vec{P}' \in \mathbb{P}$:

If $\vec{P} \neq \vec{P}'$, then $P_j \neq P'_j$ for all $j = 1, 2, 3$.

If $P'_j \leq P_j$ for some $j = 1, 2, 3$, then $P'_i \lesssim P_i$ for all $1 \leq i \leq 3$.

If we further assume that $|I_{\vec{P}'}| > 10^9|I_{\vec{P}}|$, then $P'_i \lesssim' P_i$ for all $i \neq j$.

Due to the rapid decay of coefficients over the parameters k, k', l, l' , it suffices to prove generalized restricted type estimates for models of type Λ_2 to prove generalized restricted type estimates for $T_{1_{\mathcal{R}_3}}$ with exceptional sets that are in fact allowed to depend on the dyadic shifts, tri-tile collections \mathbb{P} and \mathbb{Q} , and wave packets arising from the tri-tile collections. Hence, the discretized and localized version of Theorem 5 is

Theorem 6. Let $\sigma, \sigma' \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be shifts, and let \mathbb{P}, \mathbb{Q} be finite collections of tri-tiles with shifts σ, σ' respectively so that \mathbb{Q} is rank 1. Define the form $\Lambda_{\mathbb{P}, \mathbb{Q}}$ by

$$\Lambda_{\mathbb{P}, \mathbb{Q}}(f_1, f_2, f_3, f_4) := \sum_{\vec{P} \in \mathbb{P}} \frac{\langle f_1, \Phi_{P_1,1} \rangle \langle f_4, \Phi_{P_4,4}^{lac} \rangle \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle}{|I_{\vec{P}}|^{1/2}},$$

where the \mathbb{P} -sum is over all tri-tiles of the form $\vec{P} = (P_1, P_2, P_4)$, $\Phi_{P_1,1}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_1}$, $\Phi_{P_2,0}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_2}$, $\Phi_{P_4,4}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_4}^{lac} := I_{\vec{P}} \times [c_1|I_{\vec{P}}|^{-1}, c_2|I_{\vec{P}}|^{-1}]$ for some absolute constants $c_1 < c_2$, and

$$BHT_{\omega_{P_3}}^{\alpha}(f_2, f_3)(x) := \sum_{\vec{Q} \in \mathbb{Q}: |\omega_{\vec{Q}}| < |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,3}^{\alpha} \rangle \Phi_{Q_3,5}^{\alpha}(x),$$

where for each $\alpha \in [0, 1]$, the \mathbb{Q} -sum is over all tri-tiles of the form $\vec{Q} = (Q_1, Q_2, Q_3)$, $\Phi_{Q_1,2}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_1}$, $\Phi_{Q_2,3}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_2}$, $\Phi_{Q_3,5}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_3}$. Then $\Lambda_{\mathbb{P}, \mathbb{Q}}$ is restricted type α for all admissible tuples in $\alpha \in \mathcal{A}$, uniformly in the parameters

$$\sigma, \sigma', \mathbb{P}, \mathbb{Q}, \{\Phi_{P_i, j(i)}\}, \{\Phi_{Q_i, j(i)}^{\alpha}\}.$$

It is worth point out if $\vec{\alpha}$ has a bad index j , the restricted type estimate is *not* uniform in the sense that the major subset E_j' cannot be chosen independently of the parameters just mentioned.

As the major subset E_j' cannot be chosen independently of the wave packets $\{\Phi_{Q_i, j(i)}^{\alpha}\}$, some care must be taken in deducing Theorem 5 from Theorem 6. For this reason, we isolate the following result:

Proposition 3. To prove Theorem 5, it suffices to prove Theorem 6.

Proof. We first consider trilinear multipliers with symbols adapted to $\{\xi_1 = \xi_2 = -\xi_3/2\}$ and prove the desired L^p estimates for $T_{1_{\mathcal{R}_2}}$, namely $T_{1_{\mathcal{R}_2}} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{p_4}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p_4$, and $(p_1, p_2, p_3, p_4) \in \mathcal{A} \cap \mathcal{A}'$, where \mathcal{A} denotes the interior convex hull of the 9 points

$$\begin{aligned} A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right) \\ A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \\ A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right) \end{aligned}$$

and \mathcal{A}' denotes the interior convex hull of the collection (A'_1, \dots, A'_9) where each A'_j is gotten by the corresponding A_j by swapping the 1st and 3rd indices. By a standard discretization argument, it suffices to obtain restricted type estimates arbitrarily close to the extremal points in \mathcal{A} for the 4-form

$$\sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|} \langle f_1, \Phi_{P_1,1} \rangle \langle f_2, \Phi_{P_2,2} \rangle \langle f_3, \Phi_{P_3,3} \rangle \langle f_4, \Phi_{P_4,4} \rangle,$$

where $\mathbb{P} = (P_1, P_2, P_3, P_4)$ is a 4-tile, where each $\Phi_{P_j,j}$ is a wave packet on $P_j = (I_{\vec{P}}, \omega_{P_j})$ for $j = 1, 2, 3, 4$, for each $\vec{P} \in \mathbb{P}$, $Q_{\vec{P}} := (\omega_{P_1}, \omega_{P_2}, \omega_{P_3})$ is a Whitney cube with respect to $\{\xi_1 = \xi_2 = -\xi_3/2\}$, and $\omega_{P_4} = [c_1|I_{\vec{P}}|^{-1}, c_2|I_{\vec{P}}|^{-1}]$. Using a *BHT*-type tile decomposition, it is straightforward to obtain generalized restricted type estimates for all $\vec{\alpha}$ near the extremal points in \mathcal{A} , where the exceptional set can be taken independently of all the necessary parameters. By symmetry and fast coefficient decay, it shall therefore suffice by the proceeding discussion to prove that for every 4-tuple (E_1, E_2, E_3, E_4) of measurable subsets of \mathbb{R} , (f_1, f_2, f_3, f_4) satisfying $|f_j| \leq 1_{E_j}$ for $j = 1, 2, 3, 4$, and all $\vec{\alpha}$ in a small neighborhood of an extremal point $\vec{\beta} \in \mathcal{A}$ with bad index i , then there exists a major subset E'_i of E_i in the sense that $E'_i \subset E_i$ and $|E'_i| \geq |E_i|/2$ such that the following inequality holds for $\{f'_j\}_{j=1}^4$ where $f'_j := f_j$ if $j \neq i$ and $f'_i := f_i 1_{E'_i}$:

$$\left| \int_{\mathbb{R}} \int_0^1 T_{\vec{1}_{\mathcal{R}_3}}^{\alpha', \gamma, \gamma', \sigma, \sigma', k, k', l, l'}(f'_1, f'_2, f'_3)(x) d\alpha' f'_4(x) dx \right| \lesssim |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4},$$

where the implicit constant is uniform with respect to $\vec{\alpha}$ in a small neighborhood near but not containing the extremal point $\vec{\beta}$ and independent of $\alpha', \gamma, \gamma', \sigma, \sigma', k, k', l, l'$, and

$$\begin{aligned} & T_{\vec{1}_{\mathcal{R}_3}}^{\alpha', \gamma, \gamma', \sigma, \sigma', k, k', l, l'}(f'_1, f'_2, f'_3)(x) \\ &:= \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f'_1, \Phi_{P_1,1}^{\alpha', \gamma, k} \rangle \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}: |\omega_{\vec{Q}}| < |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f'_2, \Phi_{Q_1,2}^{\alpha, \sigma, l} \rangle \langle f'_3, \Phi_{Q_2,3}^{\alpha, \sigma', l'} \rangle \Phi_{Q_3,5}^{\alpha} d\alpha, \Phi_{P_2,0}^{\alpha', \gamma', k'} \right\rangle \overline{\Phi_{P_4,4}^{\alpha'}}(x) \\ &:= \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f'_1, \Phi_{P_1,1}^{\alpha', \gamma, k} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^d}(f_2, f_3) d\alpha, \Phi_{P_2,0}^{\alpha', \gamma', k'} \right\rangle \overline{\Phi_{P_4,4}^{\alpha'}}(x). \end{aligned}$$

By Theorem 6, we know the required generalized restricted type estimates hold. Therefore, by interpolating weak type estimates, we know $T_{\vec{1}_{\mathcal{R}_3}}$ satisfies the generalized restricted type estimate in the interior convex hull of \mathcal{A} . Moreover, by symmetry, $T_{\vec{1}_{\mathcal{R}_1}}$ must satisfy all the generalized restricted type estimates in the interior convex hull of \mathcal{A}' . Using Marcinkiewicz interpolation as discussed in [15] and then combining the L^p estimates for $T_{\vec{1}_{\mathcal{R}_1}}, T_{\vec{1}_{\mathcal{R}_2}}$ and $T_{\vec{1}_{\mathcal{R}_3}}$ yields Theorem 5. \square

5 Generalized Restricted Weak Estimates for $\Lambda_{\mathbb{P}, \mathbb{Q}}$ near A_1, A_2, A_3

5.1 Tile Decomposition

Fix dyadic shifts σ, σ' and corresponding tri-tile collections \mathbb{P} and \mathbb{Q} once and for all. By assumption, \mathbb{Q} is rank-1. Moreover, for convenience, we shall subsequently use f_j to denote f'_j for $j = 1, 2, 3, 4$ in Theorem 6 and assume by rescaling that $|E_4| = 1$ and the collections \mathbb{P}, \mathbb{Q} are sparse. Next, set

$$\tilde{\Omega} = \{M1_{E_2} \gtrsim |E_2|\} \bigcup \{M1_{E_3} \gtrsim |E_3|\}.$$

Fix $\tilde{d} \geq 0$. Then let $\mathbb{Q}^{\tilde{d}} := \left\{ \vec{Q} \in \mathbb{Q} : 1 + \frac{\text{dist}(I_{\vec{Q}}, \tilde{\Omega}^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}} \right\}$ and

$$\begin{aligned}\Omega_1^{\tilde{d}} &:= \left\{ M \left(\sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle \Phi_{Q_{3,6}}}{|I_{\vec{Q}}|^{1/2}} d\alpha \right) \gtrsim 2^{\tilde{d}} |E_2|^{1/2} |E_3|^{1/2} \right\} \\ \Omega_2^{\tilde{d}} &:= \left\{ M \left(\left[\int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \right]^2 \right) \gtrsim 2^{2\tilde{d}} |E_2| |E_3| \right\}.\end{aligned}$$

Lastly, construct

$$\Omega = \bigcup_{\tilde{d} \geq 0} \Omega_1^{\tilde{d}} \bigcup_{\tilde{d} \gtrsim 1} \Omega_2^{\tilde{d}} \bigcup \{M1_{E_1} \gtrsim |E_1|\} \bigcup \tilde{\Omega}.$$

Then for large enough implicit constants, $|\Omega| \leq 1/2$ and $\tilde{E}_4 := E_4 \cap \Omega^c$ is a major subset of E_4 since $|E_4| = 1$. Now let $\mathbb{P}^d := \left\{ \vec{P} \in \mathbb{P} : 1 + \frac{\text{dist}(I_{\vec{P}}, \Omega(\epsilon)^c)}{|I_{\vec{P}}|} \simeq 2^d \right\}$.

Assuming $|f_1| \leq 1_{E_1}, |f_2| \leq 1_{E_2}, |f_3| \leq 1_{E_3}, |f_4| \leq 1_{E_4 \cap \Omega^c}$, recall that our task in this section is to obtain the estimate $|\Lambda_{\mathbb{P}, \mathbb{Q}}(f_1, f_2, f_3, f_4)| \lesssim |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3}$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in a small neighborhood near an extremal point $\vec{\beta} \in \{A_1, A_2, A_3\}$. To decompose our tri-tiles, we shall first need to recall the following standard terminology from [12]:

It will be useful to have the following tree selection algorithm for \mathbb{P}^d essentially from [12]:

Lemma 1. Fix $d, \tilde{d} \geq 0$. Then there exist two decompositions of \mathbb{P}^d , namely $\bigcup_{n_1 \geq N_1(d)} \tilde{\mathbb{P}}_{n_1,1}^d$ and $\bigcup_{\mathfrak{d} \geq N_2(d, \tilde{d})} \mathbb{P}_{\mathfrak{d},2}^{d,\tilde{d}}$ such that $\text{Size}_1(f_1, \tilde{\mathbb{P}}_{n_1,1}^d) \lesssim 2^{-n_1}$ and $\text{Size}_2^{\tilde{d}}(f_2, f_3, \tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}_{n_1,1}^d$ and $\mathbb{P}_{\mathfrak{d},2}^{d,\tilde{d}}$ can each be written as a union of trees, i.e.

$$\begin{aligned}\tilde{\mathbb{P}}_{n_1,1}^d &= \bigcup_{T \in \mathcal{T}_{n_1,1}^d} \bigcup_{\vec{P} \in T} \vec{P} \\ \mathbb{P}_{\mathfrak{d},2}^{d,\tilde{d}} &= \bigcup_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} \bigcup_{\vec{P} \in T} \vec{P},\end{aligned}$$

such that

$$\begin{aligned}\sum_{T \in \mathcal{T}_{n_1,1}^d} |I_T| &\lesssim 2^{2n_1} \sum_{T \in \mathcal{T}_{n_1,1,*}^d} \sum_{\vec{P} \in T} |\langle f_1, \Phi_{P_{1,1}} \rangle|^2 \\ \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2,\end{aligned}$$

where $\mathcal{T}_{n_1,1,*}^d \subset \mathcal{T}_{n_1,1}^d$ and $\mathcal{T}_{\mathfrak{d},2,*}^{d,\tilde{d}} \subset \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}$, each tree in $\mathcal{T}_{n_1,1,*}^d$ is a 2-tree and each tree in $\mathcal{T}_{\mathfrak{d},2,*}^{d,\tilde{d}}$ is a 1-tree, and the collections $\mathcal{T}_{n_1,1,*}^d, \mathcal{T}_{\mathfrak{d},2,*}^{d,\tilde{d}}$ can each be written as the union of two strongly 2-disjoint subcollections. We denote this last property by

$$\begin{aligned}\mathcal{T}_{n_1,1,*}^d &= \mathcal{T}_{n_1,1,*,+}^d \bigcup \mathcal{T}_{n_1,1,*, -}^d \\ \mathcal{T}_{\mathfrak{d},2,*}^{d,\tilde{d}} &= \mathcal{T}_{\mathfrak{d},2,*,+}^{d,\tilde{d}} \bigcup \mathcal{T}_{\mathfrak{d},2,*, -}^{d,\tilde{d}}.\end{aligned}$$

Proof. We describe the procedure for producing the collection $\mathcal{T}_{n_1,1}$, as the decomposition into trees in the collection $\mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}$ is very similar. Let $N_1(d)$ be the smallest integer for which $\text{Size}_1(f_1, \mathbb{P}^d) \geq 2^{-N_1(d)}$. We may assume without loss of generality that there are only finitely many tri-tiles in the collection \mathbb{P}^d , so $N_1(d)$ exists. Assume the collection $\mathbb{P}_{m,1}^d$ has already been constructed with all the desired properties for $m < n_1$. We now

perform the following standard tile selection algorithm on the tri-tile collection $\mathbb{P}^d \cap \left[\bigcup_{N_1(d) \leq m < n_1} \tilde{\mathbb{P}}_{n_1,1}^d \right]^c$ to produce $\tilde{\mathbb{P}}_{n_1,1}^d$ with the desired properties. To this end, introduce the following notation: if P is a tile, let ξ_P denote the center of ω_P . If P and P' are tiles, we write $P' \lesssim^+ P$ if $P' \lesssim' P$ and $\xi_{P'} > \xi_P$, and $P' \lesssim^- P$ if $P' \lesssim' P$ and $\xi_{P'} < \xi_P$. Now consider the set of 2-trees in $\mathbb{P}^d \cap \left[\bigcup_{N_1(d) \leq m < n_1} \tilde{\mathbb{P}}_{n_1,1}^d \right]^c$ which are upward in the sense that

$$P_j \lesssim^+ P_{T,j} \text{ for all } \vec{P} \in T$$

and which satisfies $\sum_{\vec{P} \in T} |\langle f_1, \Phi_{P_1,1} \rangle|^2 \geq 2^{-2n_1-3} |I_T|$. If there are no trees with this property, terminate the algorithm. Otherwise, choose T among all such trees so that the center $\xi_{T,1}$ of $\omega_{P_T,1}$ is maximal and that T is maximal with respect to the set inclusion. Moreover, let T' denote that 1-tree

$$T' := \left\{ \vec{P} \in \mathbb{P}^d \cap T^c : P_1 \leq P_{T,1} \right\}.$$

Now remove T and T' from \mathbb{P}^d . Then repeat the tile selection process with the remaining tri-tiles $\mathbb{P}^d \cap (T \cup T')^c$ until there are no more upward trees satisfying the size condition. Again, by our finiteness assumption, the algorithm terminates in a finite number of steps, producing trees $T_1, T'_1, T_2, T'_2, \dots, T_M, T'_M$, where each T_j is a 2-tree and each T'_j is a 1-tree. Set

$$\begin{aligned} \mathcal{T}_{n_1,1,+}^d &= \bigcup_{j=1}^M [T_j \cup T'_j] \\ \mathcal{T}_{n_1,1,*,+}^d &= \bigcup_{j=1}^M T_j. \end{aligned}$$

The claim is now that the trees T_1, \dots, T_M are strongly 1-disjoint. Indeed, it is clear that $T_s \cap T_{s'} = \emptyset$ when $s \neq s'$. Therefore, we must have $P_1 \neq P'_1$ for all $\vec{P} \in T_s, \vec{P}' \in T_{s'}, s \neq s'$. Suppose for a contradiction that there were tri-tiles $\vec{P} \in T_s, \vec{P}' \in T_{s'}$ such that $2\omega_{P_1} \subsetneq 2\omega_{P'_1}$ and $I_{P'_1} \subset I_{T_s}$. By sparseness, we thus have $|\omega_{P'_1}| \geq 10^9 |\omega_{P_1}|$. Since $P_1 \lesssim^+ P_{T_s,1}$ and $P'_1 \lesssim^+ P_{T_{s'},1}$, we thus see that $\xi_{P_{T_{s'},1}} < \xi_{P_{T_s,1}}$. By our select algorithm, this implies $s < s'$. Also, since $|\omega_{P'_1}| \geq 10^9 |\omega_{P_1}|$, $I_{P'_1} \subset I_{T_s}$, and $P_1 \lesssim P_{T_s,1}$, it must be that $P'_1 \leq P_{T_s,1}$. Since $s < s'$, this means that $\vec{P}' \in T_s$. But T'_s and $T_{s'}$ are disjoint trees by construction, which is a contradiction. Now repeat the previous algorithm, but replace \lesssim^+ by \lesssim^- , so the trees T are downward pointing instead of upward pointing, and select the trees T so that the center $\xi_{T,j}$ is minimized rather than maximized. This yields two further collection of trees $\mathcal{T}_{n_1,1,-}^d$ and $\mathcal{T}_{n_1,1,*, -}^d$ such that for any 2-tree T consisting of unselected tiles

$$\sum_{\vec{P} \in T: P_2 \lesssim^- P_{T_2}} |\langle f_1, \Phi_{P_1,1} \rangle|^2 < 2^{-2n-3} |I_T|.$$

Letting $\mathcal{T}_{n_1,1}^d = \mathcal{T}_{n_1,1,+}^d \cup \mathcal{T}_{n_1,1,-}^d$ and $\mathcal{T}_{n_1,1,*}^d = \mathcal{T}_{n_1,1,*,+}^d \cup \mathcal{T}_{n_1,1,*, -}^d$, it follows that

$$\text{Size}_1 \left(f_1, \mathbb{P}^d \cap \left[\bigcup_{N_1(d) \leq m \leq n_1} \mathcal{T}_{m,1}^d \right]^c \right) < 2^{-2(n_1+1)}.$$

□

Now define $\mathbb{P}_{n_1,\mathfrak{d}}^{d,\tilde{d}} = \tilde{\mathbb{P}}_{n_1,1}^d \cap \tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}}$. Hence, for each $\tilde{d} \geq 0$, we may decompose the space of \mathbb{P} tri-tiles as

$$\mathbb{P} = \bigcup_{d \geq 0} \mathbb{P}^d = \bigcup_{d \geq 0} \bigcup_{n_1, \mathfrak{d}} \mathbb{P}_{n_1,\mathfrak{d}}^{d,\tilde{d}}.$$

Hence, we produce the joint decomposition $\mathbb{P} \times \mathbb{Q} = \bigcup_{\tilde{d} \geq 0} \bigcup_{d \geq 0} \mathbb{P}_{n_1,\mathfrak{d}}^{d,\tilde{d}} \times \mathbb{Q}^{\tilde{d}}$.

6 Tree Estimates

First, let T be a 2-tree consisting of bi-tiles in the collection $\mathbb{P}_{n_1, \mathfrak{d}}^{d, \tilde{d}}$. Then

$$\begin{aligned} & \left| \sum_{\tilde{P} \in T} \frac{\langle f_1, \Phi_{P_1,1} \rangle \langle f_4, \Phi_{P_4,4}^{lac} \rangle \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle}{|I_{\tilde{P}}|^{1/2}} \right| \\ & \leq \frac{(\sum_{\tilde{P} \in T} |\langle f_1, \Phi_{P_1,1} \rangle|^2)^{1/2}}{|I_T|^{1/2}} \cdot \frac{(\sum_{\tilde{P} \in T} |\langle f_4, \Phi_{P_4,4}^{lac} \rangle|^2)^{1/2}}{|I_T|^{1/2}} \cdot \sup_{\tilde{P} \in T} \left[\frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \tilde{\Phi}_{P_2,0}^\infty \right\rangle \right|}{|I_{\tilde{Q}}|} \right] |I_T| \\ & \lesssim 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} |I_T|. \end{aligned}$$

Now let $T \subset \mathbb{P}_{n_1, \mathfrak{d}}^{d, \tilde{d}}$ be a 1-tree. Then

$$\begin{aligned} & \left| \sum_{\tilde{P} \in T} \frac{\langle f_1, \Phi_{P_1} \rangle \langle f_4, \Phi_{P_4,4}^{lac} \rangle \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle}{|I_{\tilde{P}}|^{1/2}} \right| \\ & \lesssim \left[\sup_{\tilde{P} \in T} \frac{|\langle f_1, \Phi_{P_1,1} \rangle|}{|I_{\tilde{P}}|^{1/2}} \right] \left(\sum_{\tilde{P} \in T} \frac{|\langle f_4, \Phi_{P_4,4}^{lac} \rangle|^2}{|I_T|} \right)^{1/2} \cdot \left(\sum_{\tilde{P} \in T} \frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2}{|I_T|} \right)^{1/2} |I_T| \\ & \lesssim 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} |I_T|. \end{aligned}$$

Lemma 2. Let $\tilde{\mathbb{P}} \subset \mathbb{P}$ be any sub collection of tri-tiles. Then, for any $0 < \theta < 1$ and 1-tree $T \subset \tilde{\mathbb{P}}$,

$$\begin{aligned} & \left[\frac{1}{|I_T|} \sum_{\tilde{P} \in T} \left| \left\langle \int_0^1 \sum_{\tilde{Q} \in \mathbb{Q}^d: \omega_{Q_3} \supset \omega_P} \frac{1}{|I_{\tilde{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^\alpha \rangle \langle f_3, \Phi_{Q_2,3}^\alpha \rangle \Phi_{Q_3,5}^\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \right]^{1/2} \\ & \lesssim_\theta \left[\sup_{\tilde{P} \in \tilde{\mathbb{P}}} \frac{1}{|I_{\tilde{P}}|} \int 1_{E_2} \tilde{1}_{I_{\tilde{P}}} dx \right]^\theta \left[\sup_{\tilde{P} \in \tilde{\mathbb{P}}} \frac{1}{|I_{\tilde{P}}|} \int 1_{E_3} \tilde{1}_{I_{\tilde{P}}} dx \right]^{1-\theta}. \end{aligned}$$

Proof. See [12]. □

6.1 Size Restrictions

Lemma 3. Fix $d, \tilde{d}, n_1, \mathfrak{d}$ such that $\mathbb{P}_{n_1, \mathfrak{d}}^{d, \tilde{d}}$ is nonempty. Then

$$\begin{aligned} 2^{-n_1} & \lesssim 2^d |E_1| \\ 2^{-\mathfrak{d}} & \lesssim 2^{-\tilde{N}(\tilde{d}-d)} |E_2|^{1/2} |E_3|^{1/2}. \end{aligned}$$

Proof. That $2^{-n_1} \lesssim 2^d |E_1|$ is standard, so the details are omitted. Therefore, it suffices to prove $2^{-\mathfrak{d}} \lesssim 2^{-\tilde{N}(\tilde{d}-d)} |E_2|^{1/2} |E_3|^{1/2}$.

CASE 1: It clearly suffices to show that for every $\tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}$ -tree T

$$\frac{1}{|I_T|^{1/2}} \left(\sum_{\tilde{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \right)^{1/2} \lesssim 2^d |E_2|^{1/2} |E_3|^{1/2}.$$

By the triangle inequality, the LHS of the above display is at most

$$\left(\sum_{\tilde{P} \in T} \frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,3} \right\rangle \right|^2}{|I_T|} \right)^{1/2} + \left(\sum_{\tilde{P} \in T} \frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) - BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,3} \right\rangle \right|^2}{|I_T|} \right)^{1/2}.$$

Denote the first and second terms by I and II respectively. Then, by the John-Nirenberg inequality, $I \lesssim \sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \int_0^1 BHT^{\alpha, \mathbb{Q}^{\vec{d}}}(f_2, f_3) d\alpha \right| \tilde{1}_{I_{\vec{P}}} dx \lesssim 2^d 2^{\vec{d}} |E_2|^{1/2} |E_3|^{1/2}$. This is clearly acceptable by the assumption $d \geq \vec{d}$. Furthermore, $II \leq II_a + II_b$ where

$$\begin{aligned} II_a &= \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\vec{d}}: \omega_{Q_3} \supset \supset \omega_{P_2}} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \right)^{1/2} \\ II_b &= \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\vec{d}}: |\omega_{Q_3}| \simeq |\omega_{P_2}|} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \right)^{1/2}. \end{aligned}$$

Using Fubini, interchanging the integral over α and the l^2 sum, and invoking the Biest size estimate from [12] gives $II_a \lesssim_\theta 2^d |E_2|^{1/2} |E_3|^{1/2}$. To handle term II_b , first assume the $\mathbb{Q}^{\vec{d}}$ -sum is further restricted to all $|\omega_{Q_3}| = |\omega_{P_2}|$ in which case

$$\begin{aligned} II_b &\lesssim \sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \int_0^1 \sum_{\omega \in \Omega_2(T): |\omega| \leq |\omega_{P_3}|} \sum_{\vec{Q} \in \mathbb{Q}^{\vec{d}}: \omega_{Q_3} = \omega} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha d\alpha \right| \tilde{1}_{I_{\vec{P}}} dx \\ &\lesssim \int_0^1 \left[\sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \sum_{\omega \in \Omega_2(T): |\omega| \leq |\omega_{P_3}|} \sum_{\vec{Q} \in \mathbb{Q}^{\vec{d}}: \omega_{Q_3} = \omega} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} \tilde{1}_{I_{\vec{P}}} dx \right] d\alpha \\ &\lesssim 2^d |E_2|^{1/2} |E_3|^{1/2}. \end{aligned}$$

Now suppose the $\mathbb{Q}^{\vec{d}}$ -sum in II_b is restricted to the collection $\{(\vec{P}, \vec{Q}) : |\omega_{Q_3}| = 2^\lambda |\omega_{P_2}|\}$. Associate to each $\omega_{P_2} \in \Omega_2(T)$ a $(\sigma'$ -shifted) dyadic interval of length $2^\gamma |\omega_{P_2}|$ denoted by $\overline{\omega_{P_2}}$ which intersects ω_{P_2} . Of course, if $\gamma \geq 0$, there are at most two choices for $\overline{\omega_{P_2}}$, while if $\gamma < 0$ and $|\gamma| \lesssim 1$, there will be $O(1)$ choices. By the triangle inequality, it is not hard to see that we can reduce our problem to obtaining estimates for expressions of the form

$$II_b^\gamma := \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\vec{d}}: \omega_{Q_3} = \overline{\omega_{P_2}}(\gamma)} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \right)^{1/2}$$

for all $|\gamma| \lesssim 1$. Moreover, the corresponding collection $\mathbb{Q}_\gamma^{\vec{d}}(\Omega_2(T)) := \{\vec{Q} \in \mathbb{Q}^{\vec{d}} : \omega_{Q_3} = \overline{\omega_{P_2}}(\gamma)\}_{\omega_{P_2} \in \Omega_2(T)}$ can be decomposed into the union of two sets

$$\mathbb{Q}_\gamma^{\vec{d}}(\Omega_2(T)) = \mathbb{Q}_{\gamma,1}^{\vec{d}}(\Omega_2(T)) \cup \mathbb{Q}_{\gamma,2}^{\vec{d}}(\Omega_2(T))$$

where the frequencies in $\mathbb{Q}_{\gamma,1}^{\vec{d}}(\Omega_2(T))$ are lacunary in the first index and the frequencies in $\mathbb{Q}_{\gamma,2}^{\vec{d}}(\Omega_2(T))$ are lacunary in the second index. Indeed, because T is a 1-tree,

$$\text{dist}(\omega_{Q_3}, c_{\omega_{T_2}}) \leq C_1 |\omega_{Q_3}|$$

so that $\text{dist}(\omega_{Q_1}, -c_{\omega_{T_2}}) \leq C_2 |\omega_{Q_1}|$ holds for all $\vec{Q} \in \mathbb{Q}_\gamma^{\vec{d}}(\Omega_2(T))$. Now set

$$\mathbb{Q}_1^{\vec{d}}(\Omega_2(T)) = \{\vec{Q} \in \mathbb{Q}_\gamma^{\vec{d}}(\Omega_2(T)) : \text{dist}(\omega_{Q_1}, -c_{\omega_{T_2}}) \geq |\omega_{Q_1}|/2\}.$$

By construction, $\mathbb{Q}_1^{\vec{d}}(\Omega_2(T))$ is lacunary and the complement in $\mathbb{Q}_\gamma^{\vec{d}}(\Omega_2(T))$ denoted by $\mathbb{Q}_2^{\vec{d}}(\Omega_2(T))$ must be lacunary in index 2 by the rank-1 property. Hence, without loss of generality, it suffices to work only with $\mathbb{Q}_{\gamma,1}^{\vec{d}}(\Omega_2(T))$ and bound

$$\begin{aligned}
II_b^\gamma &\leq \int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} = \omega_{P_2}(\gamma)} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \right)^{1/2} d\alpha \\
&\lesssim_N \sum_{l \in \mathbb{Z}} \frac{1}{1 + |l|^N} \int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q} \in \mathbb{Q}_{\gamma,1}^{\tilde{d}}(\Omega_2(T)): I_{\vec{Q}} \subset I_T + l \cdot I_T} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle|^2 |\langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|^2}{|I_{\vec{Q}}|} \right)^{1/2} d\alpha \\
&\lesssim \sum_{l \in \mathbb{Z}} \frac{2^{d/2} |E_3|^{1/2}}{1 + |l|^N} \int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q} \in \mathbb{Q}_{\gamma,1}^{\tilde{d}}(\Omega_2(T)): I_{\vec{Q}} \subset I_T + l \cdot I_T} |\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle|^2 \right)^{1/2} d\alpha \\
&\lesssim 2^d |E_2|^{1/2} |E_3|^{1/2}.
\end{aligned}$$

CASE 2: Now assume $\tilde{d} \gg d$. It suffices to prove that for every $\mathbb{P}_{n_1, \delta}^{d, \tilde{d}}$ -tree T and some $\tilde{N} \gg 1$ that

$$\frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \right)^{1/2} \lesssim_{\tilde{N}} 2^{-\tilde{N}\tilde{d}} |E_2|^{1/2} |E_3|^{1/2}.$$

Here, we do not want to rewrite the BHT inside this sum as a difference of two other BHT s as before. Instead, we want to exploit the fact that the \vec{Q} appearing inside the \mathbb{P} -sum have finer frequency localization and so have larger time intervals $I_{\vec{Q}}$. To this end, observe that whenever $(\vec{P}, \vec{Q}) \in \mathbb{P}^d \times \mathbb{Q}^{\tilde{d}}$ satisfy $|I_{\vec{P}}| \leq |I_{\vec{Q}}|$ and $\tilde{d} \gg d$, then $\text{dist}(I_{\vec{P}}, I_{\vec{Q}}) \gtrsim 2^{\tilde{d}} |I_{\vec{Q}}|$. This is because $\vec{P} \in \mathbb{P}^d$ implies $1 + \frac{\text{dist}(I_{\vec{P}}, \Omega^c)}{|I_{\vec{P}}|} \simeq 2^d$. Therefore, using $\Omega \supset \tilde{\Omega}$,

$$\text{dist}(I_{\vec{P}}, \tilde{\Omega}^c) \lesssim 2^d |I_{\vec{P}}|.$$

If the proposed inequality did not hold, then $\text{dist}(I_{\vec{Q}}, \tilde{\Omega}^c) \leq \text{dist}(I_{\vec{Q}}, I_{\vec{P}}) + |I_{\vec{P}}| + \text{dist}(I_{\vec{P}}, \tilde{\Omega}^c) < 2^{\tilde{d}} |I_{\vec{Q}}|$, which would violate the assumption $\vec{Q} \in \mathbb{Q}^{\tilde{d}}$. With this observation, we now can write down

$$\begin{aligned}
&\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \\
&= \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} \subset \omega_{P_2}} \frac{\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi_{Q_{3,5}}^\alpha d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2 \\
&\lesssim 2^{-\tilde{N}\tilde{d}} \sum_{\omega \in \Omega\{T\}} \sum_{\vec{P} \in T: \omega_{P_2} = \omega} \left| \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} \subset \omega} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha, \tilde{1}_{I_{\vec{P}}} \right\rangle \right|^2 \\
&\lesssim 2^{-\tilde{N}\tilde{d}} \sum_{\omega \in \Omega\{T\}} \sum_{\vec{P} \in T: \omega_{P_2} = \omega} \left\| \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} \subset \omega_{P_2}} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} d\alpha \tilde{1}_{I_{\vec{Q}}} \tilde{1}_{I_T} \right\|_2^2.
\end{aligned}$$

The last line of the above display is majorized by $2^{-\tilde{N}\tilde{d}} \left\| \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2, \Phi_{Q_{1,2}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \tilde{1}_{I_T} \right\|_2^2$. Therefore,

$$\begin{aligned}
& \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle BHT_{\omega_{P_2}}^{\mathbb{Q}^{\tilde{d}}}(f_2, f_3), \Phi_{P_2,0} \right\rangle \right|^2 \right)^{1/2} \\
& \lesssim \left(2^{-\tilde{N}\tilde{d}} \sup_{\vec{P} \in \mathbb{P}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \inf_{x \in I_{\vec{P}}} M \left(\left[\int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2, \Phi_{\vec{Q}_{1,2}}^\alpha \rangle \langle f_3, \Phi_{\vec{Q}_{2,3}}^\alpha \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \right]^2 \right) (x) \right)^{1/2} \\
& \lesssim 2^{-\tilde{N}\tilde{d}/2} 2^{d/2} 2^{\tilde{d}} |E_2|^{1/2} |E_3|^{1/2} \\
& \lesssim 2^{-\tilde{N}\tilde{d}/4} |E_2|^{1/2} |E_3|^{1/2}.
\end{aligned}$$

By combining this estimate with the standard Biest size upper bound, we deduce the desired claim. \square

7 l^2 Energy Estimate

In preparation for the main semi-degenerate energy estimates, we first record a very useful inequality.

Lemma 4. Fix $\alpha \in [0, 1]$ and $\theta \in (0, 1)$. Then

$$\begin{aligned}
\left\| BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) \right\|_2 & \lesssim_\theta \left[|E_2|^{1/2} \sup_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{\int_{E_3} \tilde{1}_{I_{\vec{Q}}} dx}{|I_{\vec{Q}}|} \right]^{1-\theta} \cdot \left[|E_3|^{1/2} \sup_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{\int_{E_2} \tilde{1}_{I_{\vec{Q}}} dx}{|I_{\vec{Q}}|} \right]^\theta \\
& \lesssim 2^{\tilde{d}} |E_2|^{\frac{1+\theta}{2}} |E_3|^{\frac{2-\theta}{2}},
\end{aligned}$$

where the implicit constant in the above display can be taken independently of α .

Proof. Apply the localized BHT size/energy estimate from [12] and use the definition of $\mathbb{Q}^{\tilde{d}}$. \square

The localized energy bounds is a crucial ingredient in the proof of the following l^2 energy estimate:

Proposition 4. Fix $d, \tilde{d} \geq 0$ along with $\mathfrak{d} \geq N_2(d, \tilde{d})$. Then for any $0 < \theta < 1$,

$$\sum_{T \in \mathcal{T}_{\mathfrak{d}, 2}^{d, \tilde{d}}} |I_T| \lesssim_\theta 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}.$$

Proof. Recall from Lemma 1 that

$$\sum_{T \in \mathcal{T}_{\mathfrak{d}, 2}^{d, \tilde{d}}} |I_T| \lesssim \sum_{T \in \mathcal{T}_{\mathfrak{d}, 2, *}^{d, \tilde{d}}} |I_T| = \sum_{T \in \mathcal{T}_{\mathfrak{d}, 2, *, +}^{d, \tilde{d}}} |I_T| + \sum_{T \in \mathcal{T}_{\mathfrak{d}, 2, *, -}^{d, \tilde{d}}} |I_T|.$$

We further decompose the trees in $\mathcal{T}_{\mathfrak{d}, 2, *, +}$ into the following union:

$$\mathcal{T}_{\mathfrak{d}, 2, *, +} = \mathcal{T}_{\mathfrak{d}, 2, *, +, I} \cup \mathcal{T}_{\mathfrak{d}, 2, *, +, II}$$

where

$$\begin{aligned}
\mathcal{T}_{\mathfrak{d}, 2, *, +, I} &= \left\{ T \in \mathcal{T}_{\mathfrak{d}, 2, *, +} : \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \geq 2^{-2\mathfrak{d}-5} |I_T| \right\} \\
\mathcal{T}_{\mathfrak{d}, 2, *, +, II} &= \left\{ T \in \mathcal{T}_{\mathfrak{d}, 2, *, +} : \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT(f_2, f_3) - BHT_{\omega_P}^{\alpha, \mathbb{Q}^{\tilde{d}}} d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \geq 2^{-2\mathfrak{d}-5} |I_T| \right\}.
\end{aligned}$$

Similarly, we have the decomposition $\mathcal{T}_{\mathfrak{d},2,*, -} = \mathcal{T}_{\mathfrak{d},2,*, -, I} \cup \mathcal{T}_{\mathfrak{d},2,*, -, II}$. Therefore, putting it all together,

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, -}^{d,\tilde{d}}} |I_T| \leq \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, I}^{d,\tilde{d}}} |I_T| + \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, II}^{d,\tilde{d}}} |I_T| + \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, -, I}^{d,\tilde{d}}} |I_T| + \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, -, II}^{d,\tilde{d}}} |I_T|.$$

Using the usual BHT energy calculation,

$$\begin{aligned} \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, I}^{d,\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, I}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \\ &\lesssim 2^{2\mathfrak{d}} \left\| \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha \right\|_2^2 \\ &\lesssim 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, II}^{d,\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, +, II}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) - BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \\ &\lesssim 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}. \end{aligned}$$

The sums $\sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, -, I}^{d,\tilde{d}}} |I_T|$ and $\sum_{T \in \mathcal{T}_{\mathfrak{d},2,*, -, II}^{d,\tilde{d}}} |I_T|$ are clearly handled similarly. \square

8 l^1 Energy Boost

The standard BHT energy method involves obtaining l^2 estimates of the form

$$\sum |I_T| \lesssim 2^{2n} \sum_{T \in \mathcal{T}} \sum_{\vec{P} \in T} |\langle f, \Phi_P \rangle|^2 \lesssim 2^{2n} \|f\|_2^2,$$

where the trees $T \in \mathcal{T}$ are strongly disjoint. Because our tri-tile collection \mathbb{P} is not rank-1, we are not able to pass the analysis directly to Biest arguments. Instead, we shall need an l^1 -type energy estimate. Before stating this result precisely, we shall need to clarify terminology:

Definition 18. For each $\mathfrak{b} \geq \mathfrak{d} \geq N_2(d, \tilde{d})$, let

$$\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} = \left\{ \vec{P} \in \tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}} : \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0}^1 \right\rangle \right| \simeq 2^{-\mathfrak{b}} \right\}.$$

Theorem 7. For every $d, \tilde{d} \geq 0$, $\mathfrak{b} \geq \mathfrak{d} \geq N_2(d, \tilde{d})$ and $0 < \tilde{\epsilon} \ll 1$,

$$\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{b}}{1-\tilde{\epsilon}}} |E_2|^{1/2} |E_3|^{1/2}.$$

Before proving Theorem 7, we should explain why such an estimate is useful. Interpolating the l^2 energy bound

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim 2^{2\mathfrak{d}} |E_2|^{3/2} |E_3|^{3/2}$$

with the l^1 energy boost $\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim 2^{\sim \mathfrak{d}} |E_2|^{1/2} |E_3|^{1/2}$ ensures

$$\sum_{T \in \mathbb{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim 2^{\sim 3/2\mathfrak{d}} |E_2| |E_3|.$$

It follows that, modulo numerous details, we should have for every $0 \leq \theta_1, \theta_2 \leq 1$ such that $\theta_1 + \theta_2 = 1$

$$\begin{aligned} |\Lambda(f_1, f_2, f_3, f_4)| &\lesssim \sum_{n_1, n_4, \mathfrak{d} \geq 0} 2^{-n_1} 2^{-n_4} 2^{-\mathfrak{d}} \min \left\{ 2^{2n_1} |E_1|, 2^{\sim 3/2\mathfrak{d}} |E_2| |E_3| \right\} \\ &\lesssim \sum_{n_1, n_4, \mathfrak{d} \geq 0} 2^{-n_1(1-2\theta_1)} 2^{-n_4} 2^{-\mathfrak{d}(1-(\sim 3\theta_2\mathfrak{d}/2))} |E_1|^{\theta_1} |E_2|^{\theta_2} |E_3|^{\theta_2}. \end{aligned}$$

Choosing $\theta_1 \simeq 1/3, \theta_2 \simeq 2/3$ gives $|\Lambda(f_1, f_2, f_3, f_4)| \lesssim |E_1|^{\sim 2/3} |E_2|^{\sim 2/3} |E_3|^{\sim 2/3}$, which provides convincing evidence that $C^{1,1,-2}$ maps into $L^r(\mathbb{R})$ for all r in a small neighborhood near $\frac{1}{2}$. With this sketch in mind, it therefore remains to fill in the details.

We also record for later use

Corollary 1. *For every $d, \tilde{d} \geq 0, \mathfrak{d} \geq N_2(d, \tilde{d})$, and $0 < \tilde{\epsilon} < 1$,*

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_1|^{1/2} |E_2|^{1/2}.$$

Proof.

$$\begin{aligned} \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\vec{P},0}^2 \right\rangle \right|^2 \\ &= 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\vec{P},0} \right\rangle \right|^2 \\ &\lesssim 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} 2^{-2\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} |I_{\vec{P}}| \\ &\lesssim 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} 2^{-[2-\frac{1}{1-\tilde{\epsilon}}]\mathfrak{b}} |E_1|^{1/2} |E_2|^{1/2} \\ &\lesssim 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_1|^{1/2} |E_2|^{1/2}. \end{aligned}$$

□

Proof. [Theorem 7] Begin by using the definition of $\mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}$ and dualizing:

$$\begin{aligned} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} |I_{\vec{P}}| &\simeq 2^{\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3), \Phi_{\vec{P},0}^{\infty} \right\rangle \right| \\ &:= 2^{\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} h_{\vec{P}} \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha}(f_2, f_3) d\alpha, \tilde{1}_{I_{\vec{P}}} \right\rangle, \end{aligned}$$

where $|h_{\vec{P}}| = 1$ for all $\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}$. Rewriting the above display, we find

$$\begin{aligned} 2^{-\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} |I_{\vec{P}}| &\simeq \int_0^1 \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} \subset \omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,3}^{\alpha} \rangle \langle \Phi_{Q_3,5}^{\alpha}, h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} \rangle d\alpha \\ &= \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,1}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,2}^{\alpha} \rangle \left\langle \Phi_{Q_3,5}^{\alpha}, \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}: \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle d\alpha. \end{aligned}$$

Observe that when the tiles are restricted to a single \mathbb{Q} -tree, the sum over $\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}$ containing a frequency of the tree satisfies $\sum_{\vec{P} \in \mathbb{P}(T)} 1_{I_{\vec{P}}} \lesssim 1$. Moreover, $\left\| \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b} : \omega_{P_2} \supset \omega_{Q_3}, I_{\vec{P}} \subset I_{\vec{Q}} \text{ for some } \vec{Q} \in \mathbb{Q}^d} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \right\|_2^2 \lesssim \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}|$. Our hope is to bound $\left| \sum_{\vec{Q} \in \mathbb{Q}^d} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_{1,1}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,2}}^\alpha \rangle \left\langle \Phi_{Q_{3,5}}^\alpha, \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b} : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \right\rangle \right|$ from above by

$$|E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left[\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}| \right]^{\tilde{\epsilon}}$$

with a bound uniform in $\alpha \in [0, 1]$. Then

$$\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\mathfrak{b}} |E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left[\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}| \right]^{\tilde{\epsilon}}$$

easily implies

$$\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{b}}{1-\tilde{\epsilon}}} |E_1|^{1/2} |E_2|^{1/2}.$$

8.1 $\Lambda_{\mathbb{P}_\mathfrak{b}^\mathfrak{b}}^I(f_2, f_3)$ Estimates

Our first reduction is to handle the model

$$\begin{aligned} \Lambda_{\mathbb{P}_\mathfrak{b}^\mathfrak{b}}^I(f_1, f_2) &:= \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_{1,1}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,2}}^\alpha \rangle \left\langle \Phi_{Q_{3,5}}^\alpha, \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b} : \omega_{P_2} \supset \omega_{Q_3}, |\omega_{P_2}| \simeq |\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \right\rangle d\alpha \\ &= \int_0^1 \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} \sum_{0 \leq \lambda \leq C} \sum_{\vec{Q} \in \mathbb{Q} : \omega_{Q_3} \subset \omega_{P_2}, |\omega_{Q_3}| = 2^{-\lambda} |\omega_{P_2}|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_{1,1}}^\alpha \rangle \langle f_3, \Phi_{Q_{2,2}}^\alpha \rangle \langle \Phi_{Q_{3,5}}^\alpha, h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \rangle d\alpha. \end{aligned}$$

By the well-distributed assumption on the time intervals of $I_{\vec{Q}}$ with intersecting frequencies, we may majorize the above display by a rapidly decaying sum of expressions of the form

$$\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} \left| \langle f_2, \Phi_{\tilde{P}_{1,6}} \rangle \langle f_3, \Phi_{\tilde{P}_{2,7}} \rangle \right|,$$

where each $\{\tilde{P}_1, \vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}\}$ and $\{\tilde{P}_2 : \vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}\}$ form disjoint collections of tiles. The claim is that the above display can be bounded by an expression of the form $|E_2|^{\sim 1/2} |E_3|^{\sim 1/2} \left(\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\vec{P}}| \right)^{\tilde{\alpha}}$ for $0 < \tilde{\alpha} \ll 1$. This bound is easily achieved using the following trick.

Lemma 5. Fix a collection of disjoint tiles $\tilde{\mathbb{P}}$ and assume for all $P \in \tilde{\mathbb{P}}$, Φ_P is an L^2 -normalized wave packet on P satisfying $\text{supp } \hat{\Phi}_P \subset \omega_P$. Moreover, let $\|f\|_\infty \leq 1$. Then for every $\epsilon > 0$

$$\sum_{P \in \tilde{\mathbb{P}}} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} \lesssim_\epsilon \|f\|_2^2.$$

Proof. Construct $\tilde{\mathbb{P}}_n := \{P \in \tilde{\mathbb{P}} : |\langle f, \Phi_P^1 \rangle| \simeq 2^{-n}\}$. Hence, $\tilde{\mathbb{P}} = \bigcup_{n \geq 0} \tilde{\mathbb{P}}_n$ and

$$\begin{aligned}
\sum_{P \in \mathbb{P}} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} &= \sum_{n \geq 0} \sum_{P \in \mathbb{P}_n} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} \\
&\leq \sum_{n \geq 0} \left[\sup_{P \in \mathbb{P}_n} |\langle f, \Phi_P^1 \rangle| \right]^\epsilon \left[\sum_{P \in \mathbb{P}_n} |\langle f, \Phi_P \rangle|^2 \right] \\
&\lesssim \sum_{n \geq 0} 2^{-n\epsilon} \|f\|_2^2 \\
&\lesssim_\epsilon \|f\|_2^2.
\end{aligned}$$

□

We may now use the fact that both $\{\tilde{P}_1 : \tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}\}$ and $\{\tilde{P}_2 : \tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}\}$ are disjoint collections of tiles in order to deduce for small $\epsilon > 0$ and $\tilde{\epsilon} := \frac{\epsilon}{2+\epsilon}$ that

$$\begin{aligned}
\sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |\langle f_2, \Phi_{\tilde{P}_1,6} \rangle \langle f_3, \Phi_{\tilde{P}_2,7} \rangle| &= \sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |\langle f_2, \Phi_{\tilde{P}_1,6}^{(2+\epsilon)'} \rangle \langle f_3, \Phi_{\tilde{P}_2,7}^{(2+\epsilon)'} \rangle| \cdot |I_{\tilde{P}}|^{\tilde{\epsilon}} \\
&\leq \left(\sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |\langle f_2, \Phi_{\tilde{P}_1,6}^{(2+\epsilon)'} \rangle|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}} \left(\sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |\langle f_3, \Phi_{\tilde{P}_2,7}^{(2+\epsilon)'} \rangle|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}} \left(\sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\tilde{P}}| \right)^{\tilde{\epsilon}} \\
&\lesssim_\epsilon |E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left(\sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}} |I_{\tilde{P}}| \right)^{\tilde{\epsilon}}.
\end{aligned}$$

8.2 $\Lambda_{\mathbb{P}_\mathfrak{b}^\mathfrak{b}}^{II}(f_1, f_2)$ Estimates

Our goal is now to estimate

$$\begin{aligned}
\Lambda_{\mathbb{P}_\mathfrak{b}^\mathfrak{b}}^{II}(f_2, f_3) &= \int_0^1 \sum_{\tilde{Q} \in \mathbb{Q}} \frac{1}{|I_{\tilde{Q}}|^{1/2}} \langle f_2, \Phi_{\tilde{Q}_1,2}^\alpha \rangle \langle f_3, \Phi_{\tilde{Q}_2,3}^\alpha \rangle \left\langle \Phi_{\tilde{Q}_3,5}^\alpha, \sum_{\tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b} : \omega_{P_2} \supset \omega_{Q_3}, |\omega_{P_2}| > |\omega_{Q_3}|} h_{\tilde{P}} \Phi_{\tilde{P}}^\infty \right\rangle d\alpha \\
&:= \int_0^1 \Lambda_{\mathbb{P}_\mathfrak{b}^\mathfrak{b}}^{II,\alpha}(f_2, f_3) d\alpha.
\end{aligned}$$

To this end, we now consider $\alpha \in [0, 1]$ to be a fixed and compute the 3rd index size and energy.

8.3 3-Size Bounds

Fix a $\mathbb{Q}^{\tilde{d}}$ -tree T overlapping in either the 1st or 2nd index. Observe that since each $\eta_{\omega_{P_2}}$ can be chosen so that $\text{supp } \hat{\eta}_{\omega_{P_2}} \subset [c_{\omega_{P_2}} - \frac{19}{40}|\omega_{P_2}|, c_{\omega_{P_2}} + \frac{19}{40}|\omega_{P_2}|]$, the collection

$$\mathbb{P}_\mathfrak{b}^\mathfrak{b}(T) := \left\{ \tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b} : \exists \tilde{Q} \in T, \langle \Phi_{\tilde{Q}_3,5}^\alpha, \Phi_{\tilde{P}}^\infty * \eta_{\omega_{P_2}} \rangle \neq 0, |\omega_{P_2}| > |\omega_{Q_3}| \right\}$$

consists of tiles with disjoint time projections. Indeed, it is easy to check that for large enough implicit constant (depending on the implicit constants in the definition of a tree), every tile $P_2 : \tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}(T)$ must contain the $\mathbb{Q}^{\tilde{d}}$ -tree's top frequency band ω_{T_3} , and $\mathbb{P}_\mathfrak{b}^\mathfrak{b}$ consists of disjoint tri-tiles. By the frequency restriction $\omega_{P_2} \supset \omega_{Q_3}$ and disjointness of the tiles $\{P_2 : \tilde{P} \in \mathbb{P}_\mathfrak{b}^\mathfrak{b}\}$, we observe

$$\begin{aligned}
& \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3,5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_0^b: \omega_{P_2} \supset \omega_{Q_3}, |\omega_{Q_3}| > |\omega_{P_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} \right\rangle \right|^2 \right)^{1/2} \\
&= \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3,5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_0^b(T): |\omega_{P_2}| > |\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle \right|^2 \right)^{1/2} \\
&\lesssim \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3,5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_0^b(T)} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle \right|^2 \right)^{1/2} \\
&+ \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3,5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_0^b(T): |\omega_{P_2}| \simeq |\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle \right|^2 \right)^{1/2} \\
&:= I + II.
\end{aligned}$$

For the last line, it is straightforward to check that for large enough implicit constant

$$\left\{ \vec{P} \in \mathbb{P}_0^b(T) : \exists \vec{Q} \in T, |\omega_{P_2}| < |\omega_{Q_3}|, \langle \Phi_{Q_3,5}^\alpha, \Phi_{\vec{P}}^\infty * \eta_{\omega_{P_2}} \rangle \neq 0 \right\} = \emptyset.$$

Our goal is then to show $I, II \lesssim 1$, in which case $2^{-n_3} \lesssim 1$. Term I may be estimated

$$\begin{aligned}
I &\lesssim \sum_{l \in \mathbb{Z}} \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3,5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_0^b(T): I_{\vec{P}} \subset I_T + l} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle \right|^2 \right)^{1/2} \\
&= \sum_{l \in \mathbb{Z}} \left(\frac{1}{|I_T|} \int_{\mathbb{R}} \left| \sum_{\vec{P} \in \mathbb{P}_0^b(T): I_{\vec{P}} \subset I_T + l} \tilde{1}_{I_{\vec{P}}} \right|^2 \tilde{1}_{I_T} dx \right)^{1/2}.
\end{aligned}$$

It clearly suffices to prove $\left(\frac{1}{|I_T|} \int_{\mathbb{R}} \left| \sum_{\vec{P} \in \mathbb{P}_0^b(T): I_{\vec{P}} \subset I_T + l} \tilde{1}_{I_{\vec{P}}} \right|^2 \tilde{1}_{I_T} dx \right)^{1/2} \lesssim \frac{1}{1+l^N}$ for each $l \in \mathbb{Z}$ and some $N \geq 2$.

Lemma 6. *Let $\tilde{1}_{I_{Q_1}}$ and $\tilde{1}_{I_{Q_2}}$ be two rapidly decaying bump functions adapted to dyadic intervals I_{Q_1} and I_{Q_2} respectively (decaying at some polynomial rate \tilde{N} away from $c_{I_{Q_1}}$ and $c_{I_{Q_2}}$). Then whenever $|I_{Q_2}| \geq |I_{Q_1}|$, there exists a rapidly decaying function $\tilde{1}_{I_{Q_2}}^N$ (decaying like $1/|x|^N$ away from $c_{I_{Q_2}}$) for some $1 < N \lesssim \tilde{N}$ such that*

$$\int_{\mathbb{R}} \tilde{1}_{I_{Q_1}} \tilde{1}_{I_{Q_2}} dx \lesssim_N \frac{|I_{Q_1}|}{1 + \left[\frac{\text{dist}(I_{Q_1}, I_{Q_2})}{|I_{Q_2}|} \right]^N} \lesssim \int_{\mathbb{R}} 1_{I_{Q_1}} \tilde{1}_{I_{Q_2}}^N dx.$$

Proof. The proof is straightforward and is therefore omitted. \square

Lemma 7.

$$\frac{1}{|I_T|} \int_{\mathbb{R}} \left| \sum_{\vec{P} \in \mathbb{P}_0^b(T): I_{\vec{P}} \subset I_T + l} \tilde{1}_{I_{\vec{P}}} \right|^2 \tilde{1}_{I_T} dx \lesssim \frac{1}{1+l^N}.$$

Proof.

$$\begin{aligned}
\frac{1}{|I_T|} \int_{\mathbb{R}} \left| \sum_{\vec{P} \in \mathbb{P}_\delta^b(T): I_{\vec{P}} \subset I_T + l|I_T|} \tilde{1}_{I_{\vec{P}}} \right|^2 \tilde{1}_{I_T} dx &\lesssim \frac{1}{1+l^N} \frac{1}{|I_T|} \int_{\mathbb{R}} \sum_{\vec{P}_1, \vec{P}_2 \in \mathbb{P}_\delta^b(T): I_{P_1}, I_{P_2} \subset I_T + l|I_T|} \tilde{1}_{I_{P_1}} \tilde{1}_{I_{P_2}} dx \\
&\lesssim \frac{1}{1+l^N} \frac{1}{|I_T|} \int_{\mathbb{R}} \sum_{P_1, P_2 \in \mathbb{P}_\delta^b(T): |I_{P_2}| \geq |I_{P_1}|: I_{P_1}, I_{P_2} \subset I_T + l|I_T|} \tilde{1}_{I_{P_1}} \tilde{1}_{I_{P_2}} dx \\
&+ \frac{1}{1+l^N} \frac{1}{|I_T|} \int_{\mathbb{R}} \sum_{P_1, P_2 \in \mathbb{P}_\delta^b(T): |I_{P_2}| < |I_{P_1}|: I_{P_1}, I_{P_2} \subset I_T + l|I_T|} \tilde{1}_{I_{P_1}} \tilde{1}_{I_{P_2}} dx \\
&\lesssim \frac{1}{1+l^N} \frac{1}{|I_T|} \int_{\mathbb{R}} \sum_{P_2 \in \mathbb{P}_\delta^b(T): I_{P_2} \subset I_T + l|I_T|} \tilde{1}_{I_{P_2}}^N dx \\
&+ \frac{1}{1+l^N} \frac{1}{|I_T|} \int_{\mathbb{R}} \sum_{P_1 \in \mathbb{P}_\delta^b(T): I_{P_1} \subset I_T + l|I_T|} \tilde{1}_{I_{P_1}}^N dx \\
&\lesssim \frac{1}{1+l^N}.
\end{aligned}$$

□

8.3.1 Term II

For II, it suffices to observe

$$II \leq \sum_{|k| \leq C} \left(\frac{1}{|I_T|} \sum_{\vec{Q} \in T} \left| \left\langle \Phi_{Q_3, 5}^\alpha, \sum_{\vec{P} \in \mathbb{P}_\delta^b(T): |\omega_{P_2}| = 2^k |\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \right\rangle \right|^2 \right)^{1/2}.$$

Therefore, for each scale, we compute

$$\sum_{\vec{P} \in T: |\omega_P| = 2^\lambda} \left| \left\langle \Phi_{P_3}^\alpha, \sum_{\vec{P} \in \mathbb{P}_\delta^b(T): |\omega_{P_2}| = 2^k |\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}} \right\rangle \right|^2 \lesssim \left\| \left[\sum_{\vec{Q} \in \mathbb{Q}_T: |\omega_Q| = 2^{k+\lambda}} 1_{I_{\vec{P}}} \right] \tilde{1}_{I_T} \right\|_2^2.$$

Summing over all $\lambda \in \mathbb{Z}$ yields

$$\begin{aligned}
II &\lesssim \frac{1}{|I_T|^{1/2}} \sum_{|k| \leq C} \left\| \left(\sum_{\lambda \in \mathbb{Z}} \left| \sum_{\vec{P} \in \mathbb{P}_\delta^b(T): |\omega_{P_2}| = 2^{k+\lambda}} 1_{I_{\vec{P}}} \right|^2 \right)^{1/2} \tilde{1}_{I_T} \right\|_2 \\
&\lesssim \frac{1}{|I_T|^{1/2}} \left\| \left(\sum_{\lambda \in \mathbb{Z}} \left[\sum_{\vec{P} \in \mathbb{P}_\delta^b(T): |\omega_{P_2}| = 2^\lambda} 1_{I_{\vec{P}}} \right]^2 \right)^{1/2} \tilde{1}_{I_T} \right\|_2 \lesssim 1.
\end{aligned}$$

8.4 3-Energy Bound

Letting $\Phi_{\vec{Q}}^\infty := \tilde{1}_{I_{\vec{Q}}} * \eta_{\omega_Q}$ and

$$c_{Q_3} := 2^{-n_3} \overline{\left\langle \sum_{\vec{P} \in \mathbb{P}_\delta^b: \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}}, \Phi_{Q_3, 5}^\alpha \right\rangle} \cdot \left[\sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T} \left| \left\langle \sum_{\vec{P} \in \mathbb{P}_\delta^b: \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}}, \Phi_{Q_3, 5}^\alpha \right\rangle \right|^2 \right]^{-1/2}$$

for all $\vec{Q} \in \bigcup_{T \in \mathbb{T}} T$, it follows that

$$\begin{aligned}
E_3 &\simeq 2^{-n_3} \left[\sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T} \left| \left\langle \sum_{\vec{P} \in \mathbb{P}_\circ^b : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}}, \Phi_{Q_3,5}^\alpha \right\rangle \right|^2 \right]^{1/2} \\
&= \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T} \left\langle \sum_{\vec{P} \in \mathbb{P}_\circ^b : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}}, c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle \\
&= \sum_{\vec{P} \in \mathbb{P}_\circ^b} \left\langle h_{\vec{P}} \Phi_{\vec{P}}^\infty, \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T : \omega_{Q_3} \subset \omega_{P_2}} c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle \\
&= \left\langle \sum_{\vec{P} \in \mathbb{P}_\circ^b} h_{\vec{P}} \Phi_{\vec{P}}^\infty, \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T} c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle + \sum_{\vec{P} \in \mathbb{P}_\circ^b} \left\langle h_{\vec{P}} \Phi_{\vec{P}}^\infty, \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T : \omega_{Q_3} \supseteq \omega_{P_2}} c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle \\
&:= E_3^I + E_3^{II}.
\end{aligned}$$

From the definition, we have for all $\vec{Q} \in \bigcup_{T \in \mathbb{T}} T$

$$|c_{Q_3}| \simeq \left| \left\langle \sum_{\vec{P} \in \mathbb{P}_\circ^b : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}}, \Phi_{Q_3,5}^\alpha \right\rangle \right| \left[\sum_{T \in \mathbb{T}} |I_T| \right]^{-1/2}$$

and for any subtree $T' \subset T$, $\sum_{\vec{Q} \in T'} |c_{Q_3}|^2 \lesssim \frac{|I_{T'}|}{\sum_{T \in \mathbb{T}} |I_T|}$.

Lemma 8. Assume $|h_{\vec{P}}| \leq 1$ for all $\vec{P} \in \mathbb{P}_\circ^b$. Then

$$\left\| \sum_{\vec{P} \in \mathbb{P}_\circ^b} h_{\vec{P}} \Phi_{\vec{P}}^\infty \right\|_2 \lesssim \left(\sum_{\vec{P} \in \mathbb{P}_\circ^b} |I_{\vec{P}}| \right)^{1/2}.$$

Proof. It suffices to note

$$\begin{aligned}
\left\| \sum_{\vec{P} \in \mathbb{P}_\circ^b} h_{\vec{P}} \Phi_{\vec{P}}^\infty \right\|_2^2 &= \sum_{\vec{P}, \tilde{\vec{P}} \in \mathbb{P}_\circ^b} h_{\vec{P}} h_{\tilde{\vec{P}}} \langle \Phi_{\vec{P}}^\infty, \Phi_{\tilde{\vec{P}}}^\infty \rangle \\
&= \left(\sum_{|I_{\vec{P}}| > |I_{\tilde{\vec{P}}}|} + \sum_{|I_{\vec{P}}| \simeq |I_{\tilde{\vec{P}}}|} + \sum_{|I_{\vec{P}}| < |I_{\tilde{\vec{P}}}|} \right) h_{\vec{P}} h_{\tilde{\vec{P}}} \langle \Phi_{\vec{P}}^\infty, \Phi_{\tilde{\vec{P}}}^\infty \rangle \\
&= I + II + III.
\end{aligned}$$

It is straightforward to observe $|II| \lesssim \sum_{\vec{P} \in \mathbb{P}_\circ^b} |I_{\vec{P}}|$. By symmetry, it suffices to handle term I and compute

$$|I| \leq \sum_{\vec{P} \in \mathbb{P}_\circ^b} \left[\sum_{\tilde{\vec{P}} \in \mathbb{P}_\circ^b : \omega_{\tilde{\vec{P}}} \supset \omega_{\vec{P}}} \left| \langle \Phi_{\vec{P}}^\infty, \Phi_{\tilde{\vec{P}}}^\infty \rangle \right| \right] \lesssim \sum_{\vec{P} \in \mathbb{P}_\circ^b} |I_{\vec{P}}|.$$

□

8.5 E_3^I Estimate

Using the fact that the trees $T \in \mathbb{T}$ form a strongly disjoint collection and that for any subtree $T' \subset T \in \mathbb{T}$, $\sum_{\vec{Q} \in T'} |c_{Q_3}|^2 \lesssim \frac{|I_{T'}|}{\sum_{T \in \mathbb{T}} |I_T|}$, we may deploy the standard BHT energy estimate from [12], say, to deduce

$$E_3^I = \left| \left\langle \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} h_{\vec{P}} \Phi_{\vec{P}}^\infty, \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T} c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle \right| \lesssim \left\| \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} h_{\vec{P}} \Phi_{\vec{P}}^\infty \right\|_2 \lesssim \left(\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}| \right)^{1/2}.$$

Hence, it remains to obtain satisfactory bounds on E_3^{II} .

8.6 E_3^{II} Estimate

Now note

$$\begin{aligned} E_3^{II} &\lesssim \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} \left| \left\langle h_{\vec{P}} \Phi_{\vec{P}}^\infty, \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T: \omega_{Q_3} \supseteq \omega_{P_2}} c_{Q_3} \Phi_{Q_3,5}^\alpha \right\rangle \right| \\ &\lesssim \frac{\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|}{(\sum_{T \in \mathbb{T}} |I_T|)^{1/2}} \left[\sup_{\vec{P} \in \mathbb{P}_\mathfrak{b}} \left\| \sum_{T \in \mathbb{T}} \sum_{\vec{Q} \in T: \omega_{Q_3} \supseteq \omega_{P_2}} 1_{I_{\vec{Q}}} \right\|_{L^\infty(\mathbb{R})} \right] \\ &\lesssim \frac{\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|}{(\sum_{T \in \mathbb{T}} |I_T|)^{1/2}}. \end{aligned}$$

Putting our estimates for E_3^I and E_3^{II} together yields

$$E_3 = 2^{-n_3} \left(\sum_{T \in \mathbb{T}} |I_T| \right)^{1/2} \simeq E_3^I + E_3^{II} \lesssim \left(\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}| \right)^{1/2} + \frac{\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|}{(\sum_{T \in \mathbb{T}} |I_T|)^{1/2}}.$$

CASE 1: $\sum_{T \in \mathbb{T}} |I_T| < \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|$. Then $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{n_3} \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|$.

CASE 2: $\sum_{T \in \mathbb{T}} |I_T| \geq \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|$. Then $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2n_3} \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|$.

Recall the 3-size estimate $2^{-n_3} \lesssim 1$ so that in either case $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2n_3} \sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}|$ and $E_3 \lesssim \left(\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}| \right)^{1/2}$. Moreover, $S_1, S_2, S_3 \lesssim 1$, $E_1 \lesssim |E_2|^{1/2}, E_2 \lesssim |E_3|^{1/2}$, where S_1, S_2, S_3 and E_1 and E_2 are the usual sizes and energies from [12]. Using the fundamental Size-Energy inequality, we have for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ such that $\theta_1 + \theta_2 + \theta_3 = 1$

$$\left| \Lambda_{\mathbb{P}_\mathfrak{b}}^{II}(f_2, f_3) \right| \lesssim \prod_{j=1}^3 E_j^{1-\theta_j} S_j^{\theta_j}.$$

Therefore, we may pick $\theta_1, \theta_2 = \tilde{\epsilon}$ and $\theta_3 = 1 - 2\tilde{\epsilon}$ to deduce the claim

$$\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}| \lesssim 2^{\mathfrak{b}} |E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left[\sum_{\vec{P} \in \mathbb{P}_\mathfrak{b}} |I_{\vec{P}}| \right]^{\tilde{\epsilon}}.$$

9 Generalized Restricted Weak Estimates near A_1, A_2, A_3

We now combine the proceeding results to prove generalized restricted type estimates uniform in small neighborhoods near each point in $\{A_1, A_2, A_3\}$. The decomposition

$$\mathbb{P} \times \mathbb{Q} = \bigcup_{d \geq 0} \bigcup_{\tilde{d} \geq 0} \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}} \times \mathbb{Q}^{\tilde{d}}$$

enables us to rewrite $\Lambda_{\mathbb{P}, \mathbb{Q}}(f_1, f_2, f_3, f_4)$ as

$$\sum_{d, \tilde{d} \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{\mathfrak{d} \geq N_2(d, \tilde{d})} \sum_{\tilde{P} \in \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle$$

For fixed $d, \tilde{d} \geq 0, n_1 \geq N_1(d)$ and $\mathfrak{d} \geq N_2(d, \tilde{d})$, we may further rewrite

$$\begin{aligned} & \sum_{\tilde{P} \in \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle \\ &= \sum_{T \in \mathcal{T}_{n_1, 1}^d} \sum_{\tilde{P} \in T \cap \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle \\ &= \sum_{T \in \mathcal{T}_{\mathfrak{d}, 2}^{d, \tilde{d}}} \sum_{\tilde{P} \in T \cap \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle. \end{aligned}$$

Each tree in $\mathcal{T}_{n_1, 1}^d$ is overlapping in either the 1st or 2nd index. Using the tree and energy estimates gives

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_{n_1, 1}^d} \sum_{\tilde{P} \in T \cap \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle \right| \\ & \lesssim 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \sum_{T \in \mathcal{T}_{n_1, 1}^d} |I_T| \\ & \lesssim 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} [2^{2n_1} |E_1|]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}_{\mathfrak{d}, 2}^{d, \tilde{d}}} \sum_{\tilde{P} \in T \cap \tilde{\mathbb{P}}_{n_1, \mathfrak{d}}^{d, \tilde{d}}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \langle f'_1, \Phi_{\tilde{P}, 1}^{\alpha', \gamma, k} \rangle \langle f_4, \Phi_{\tilde{P}, 4}^{\alpha'} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{\tilde{P}, 0}^{\alpha', \gamma', k'} \right\rangle \right| \\ & \lesssim 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \left[\sum_{T \in \mathcal{T}_{\mathfrak{d}, 2}^{d, \tilde{d}}} |I_T| \right] \\ & \lesssim_{\theta} 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \min \left\{ 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}, 2^{\frac{2}{1-\varepsilon}} |E_2|^{\frac{1}{2}} |E_3|^{\frac{1}{2}} \right\}. \end{aligned}$$

Hence, for any $(\theta_1, \theta_2, \theta_3)$ satisfying $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$ one has

$$\begin{aligned} & |\Lambda_{\mathbb{P}, \mathbb{Q}}(f_1, f_2, f_3, f_4 1_{\Omega'})| \\ & \lesssim \sum_{\tilde{d}, d \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{\mathfrak{d} \geq N_2(d, \tilde{d})} 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \min \left\{ 2^{2n_1} |E_1|, 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}, 2^{\frac{2}{1-\varepsilon}} |E_2|^{\frac{1}{2}} |E_3|^{\frac{1}{2}} \right\} \\ & \lesssim \sum_{\tilde{d}, d \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{\mathfrak{d} \geq N_2(d, \tilde{d})} 2^{-\tilde{N}d} 2^{2\theta_2 \tilde{d}} 2^{-n_1} 2^{-\mathfrak{d}} 2^{2\theta_1 n_1} |E_1|^{\theta_1} 2^{2\theta_2 \mathfrak{d}} |E_2|^{\theta_2(1+\theta)} |E_3|^{\theta_2(2-\theta)} 2^{\theta_3 \frac{\mathfrak{d}}{1-\varepsilon}} |E_2|^{\frac{\theta_3}{2}} |E_3|^{\frac{\theta_3}{2}} \\ & = \sum_{\tilde{d}, d \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{\mathfrak{d} \geq N_2(d, \tilde{d})} 2^{-\tilde{N}d} 2^{2\theta_2 \tilde{d}} 2^{-n_1(1-2\theta_1)} 2^{-\mathfrak{d}[1-2\theta_2-\frac{\theta_3}{1-\varepsilon}]} |E_1|^{\theta_1} |E_2|^{\theta_2(1+\theta)+\frac{\theta_3}{2}} |E_3|^{\theta_2(2-\theta)+\frac{\theta_3}{2}}. \end{aligned}$$

Take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $A_1 = (1, \frac{1}{2}, \frac{1}{2}, -1)$, set $\theta = \frac{1}{2}, \theta_1 \simeq 0, \theta_2 \simeq 0, \theta_3 \simeq 1$. For $A_2 = (\frac{1}{2}, \frac{1}{2}, 1, -1)$, set $\theta \simeq 0, \theta_1 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$. Lastly, for $A_3 = (\frac{1}{2}, 1, \frac{1}{2} - 1)$, set $\theta \simeq 1, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$.

10 Generalized Restricted Weak Estimates near A_4 and A_5

By rescaling, we may assume $|E_1| = 1$. Construct the exceptional set

$$\tilde{\Omega} = \{M1_{E_2} \gtrsim |E_2|\} \cup \{M1_{E_3} \gtrsim |E_3|\}.$$

Let $\mathbb{Q}^{\tilde{d}} := \left\{ \tilde{Q} \in \mathbb{Q} : 1 + \frac{\text{dist}(I_{\tilde{Q}}, \tilde{\Omega}^c)}{|I_{\tilde{Q}}|} \simeq 2^{\tilde{d}} \right\}$ and define

$$\begin{aligned} \Omega_1^{\tilde{d}} &= \left\{ M \left[\int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha \right] \gtrsim_{\theta} 2^{\tilde{d}} |E_2|^{\theta} |E_3|^{1-\theta} \right\} \\ \Omega_2^{\tilde{d}} &= \left\{ M \left(\left[\int_0^1 \sum_{\tilde{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2, \Phi_{Q_1, 2}^{\alpha} \rangle \langle f_3, \Phi_{Q_2, 3}^{\alpha} \rangle|}{|I_{\tilde{Q}}|} \tilde{1}_{I_{\tilde{Q}}} d\alpha \right]^2 \right) \gtrsim_{\theta} 2^{2\tilde{d}} |E_2| |E_3| \right\}. \end{aligned}$$

Lastly, set

$$\Omega = \bigcup_{\tilde{d} \geq 0} \Omega_1^{\tilde{d}} \bigcup_{\tilde{d} \gtrsim 1} \Omega_2^{\tilde{d}} \bigcup \{M1_{E_4} \gtrsim |E_4|\} \bigcup \tilde{\Omega}.$$

Then for large enough implicit constants, $|\Omega| \leq 1/2$ and $\tilde{E}_1 := E_1 \cap \Omega^c$ is a major subset of E_1 since $|E_1| = 1$. The rest of the proof of Theorem 6 near $\{A_4, A_5\}$ proceeds exactly as before. As the end of the day, we have

$$\begin{aligned} & |\Lambda_{\mathbb{P}, \mathbb{Q}}(f_1 1_{\Omega'}, f_2, f_3, f_4)| \\ & \lesssim_{\tilde{\epsilon}} \sum_{\tilde{d}, d \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{\mathfrak{d} \geq N_2(d, \tilde{d})} 2^{-\tilde{N}d} 2^{2\theta_2 \tilde{d}} 2^{-n_1(1-2\theta_1)} 2^{-\mathfrak{d}[1-2\theta_2 - \frac{\theta_3}{1-\tilde{\epsilon}}]} |E_2|^{\theta_2(1+\theta) + \frac{\theta_3}{2}} |E_3|^{\theta_2(2-\theta) + \frac{\theta_3}{2}} |E_4|. \end{aligned}$$

Take $\tilde{\epsilon} \simeq 0$. We may then set $\theta \simeq 0, \theta_1 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}$ and $\theta_3 \simeq 0$ to deduce the desired estimates near $A_4 = (-\frac{3}{2}, \frac{1}{2}, 1, 1)$ and $\theta \simeq 0, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq \frac{1}{2}, \theta_3 \simeq 0$ to deduce the estimates near $A_5 = (-\frac{3}{2}, 1, \frac{1}{2}, 1)$.

11 Generalized Restricted Weak Estimates near A_6, A_7, A_8, A_9

Recall that the trilinear simplex multiplier

$$\tilde{C}^{1,1,-1/2} : (f_1, f_2, f_3) \mapsto \int_{\xi_1 < \xi_2 < -\xi_3/2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3$$

satisfies the identity

$$\tilde{C}^{1,1,-1/2}(f_1, f_2, f_3)(x) = \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - H^+(\tilde{C}^{1,1}(f_1, f_2) \cdot f_3)(x) - H^-(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x).$$

Therefore, the 3-adjoint denoted by $\tilde{C}_{*,3}^{1,1,-1/2}$ defined by the usual property

$$\int_{\mathbb{R}} \tilde{C}^{1,1,-1/2}(f_1, f_2, f_4)(x) f_3(x) dx = \int_{\mathbb{R}} \tilde{C}_{*,3}^{1,1,-1/2}(f_1, f_2, f_3)(x) f_4(x) dx$$

for all $(f_1, f_2, f_3, f_4) \in \mathcal{S}(\mathbb{R})^4$ is writable as

$$\tilde{C}_{*,3}^{1,1,-1/2}(f_1, f_2, f_3)(x) = \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - \tilde{C}^{1,1}(f_1, f_2)(x) \cdot H^-(f_3)(x) - \tilde{C}^{1,1}(f_1(-\cdot) \cdot H^+(f_3)(-\cdot), f_2)(-x).$$

Indeed, it suffices to check the last term, which we claim is the 3-adjoint of the map $(f_1, f_2, f_3) \mapsto -H^-(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x)$. Indeed, we have

$$\begin{aligned} \int_{\mathbb{R}} H^- \left[f_1 \cdot \tilde{C}^{-1/2,1}(f_4, f_2) \right] (x) f_3(x) dx &= \left\langle H^- \left[f_1 \cdot \tilde{C}^{-1/2,1}(f_4, f_2) \right], \bar{f}_3 \right\rangle \\ &= \left\langle \tilde{C}^{-1/2,1}(f_4, f_2), \bar{f}_1 \cdot H^- [\bar{f}_3] \right\rangle \\ &= \left\langle \mathcal{F} \left[\tilde{C}^{-1/2,1}(f_4, f_2) \right], \mathcal{F} [\bar{f}_1 \cdot H^- [\bar{f}_3]] \right\rangle \\ &= \left\langle \mathcal{F} \left[\tilde{C}^{1,1} \left(\mathcal{F}^{-1} \left[\overline{\mathcal{F} [\bar{f}_1 \cdot H^- [\bar{f}_3]]} \right] \right), f_2 \right], \overline{\mathcal{F}(f_4)} \right\rangle \\ &= \left\langle \tilde{C}^{1,1} \left(\mathcal{F}^{-1} \left[\overline{\mathcal{F} [\bar{f}_1 \cdot H^- [\bar{f}_3]]} \right], f_2 \right), \mathcal{F}^{-1} [\overline{\mathcal{F}(f_4)}] \right\rangle \\ &= \left\langle \tilde{C}^{1,1} (f_1(-\cdot) \cdot H^+ [f_3](-\cdot), f_2), \bar{f}_4(-\cdot) \right\rangle \\ &= \int_{\mathbb{R}} \tilde{C}^{1,1} (f_1(-\cdot) \cdot H^+ [f_3](-\cdot), f_2)(-x) f_4(x) dx. \end{aligned}$$

Hence, using the *BHT* and Hilbert transform estimates, $\tilde{C}_{*,3}$ maps into $L^r(\mathbb{R})$ for all $r \in (\frac{2}{3}, \infty)$. Therefore, generically speaking, we should not expect the adjoint models to map below $L^{\frac{2}{3}}(\mathbb{R})$. We now proceed to prove the generalized restricted type estimates near the points A_6, A_7, A_8, A_9 , where the adjoint index is restricted to map into the above range. By symmetry, it will suffice to prove the estimate only near the points $A_8 = (0, 1, -\frac{1}{2}, \frac{1}{2})$ and $A_9 = (\frac{1}{2}, 1 - \frac{1}{2}, 0)$, for which 3 is the adjoint index. Indeed, estimates near A_6, A_7 are obtained from estimates near A_8 and A_9 by interchanging the roles of f_2, f_3 .

The adjoint situation is more complicated in the semi-degenerate case than in the fully non-degenerate one because one cannot simply flip the frequency inclusions to reduce to the situation where the exceptional set is associated with functions in the 2nd and 3rd index. This is ultimately because the paracomposition

$$\sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1, \Phi_{P_1,1} \rangle \langle f_4, \Phi_{P_4,4}^{lac} \rangle \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle$$

satisfies no restricted type estimates because \mathbb{P} is not a rank-1 collection of ti-tiles. Moreover, if one tries to repeat the arguments for A_1, A_2, A_3, A_4, A_5 estimates, one cannot enlarge the exceptional set Ω to obtain good control over the averages of the $BHT^{\mathbb{Q}}$ -type operators on intervals much smaller than the time-lengths of the tiles in the \mathbb{P} -tree T , which may be much farther from Ω^c than the time-lengths of the corresponding \mathbb{Q} -tiles. The way around this obstruction is to decompose our collection of degenerate tri-tiles \mathbb{P} . To motivate our construction, we first focus on estimating standard tree sizes of the form

$$S(f_2, f_3, T) := \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T} \left| \left\langle \sum_{\vec{Q}: \omega_{Q_3} \subset \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2}.$$

So, fix a \mathbb{P} -tree T . Suppose for every $\vec{P} \in T$

$$I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 1 \right\}^c \neq \emptyset.$$

Then, the John-Nirenberg inequality combined with the Biest size estimate implies $S(f_2, f_3, T) \lesssim 1$. Define

$$\mathbb{P}_1 = \left\{ \vec{P} \in \mathbb{P} : I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 1 \right\}^c \neq \emptyset \right\}.$$

Next let

$$\mathbb{P}_2 = \left\{ \vec{P} \in \mathbb{P} \cap \mathbb{P}_1^c : I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^1 \right\}^c \neq \emptyset \right\}.$$

Inductively construct

$$\mathbb{P}_k = \left\{ \vec{P} \in \mathbb{P} \cap \left(\bigcup_{\vec{k} \leq k-1} \mathbb{P}_{\vec{k}} \right)^c : I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^k \right\}^c \neq \emptyset \right\}.$$

By design, any \mathbb{P} -tree T satisfies $T = \bigcup_{k \in \mathbb{N}} T \cap \mathbb{P}_k$ and $\left| \bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \right| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$.

Lemma 9. *Let $T \in \mathbb{T}$ be a tree of lacunary tiles. Then the following adjoint size estimate holds:*

$$\begin{aligned} & \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\omega_{Q_3} \subset \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ & \lesssim \left[\sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{E_2} \tilde{1}_{I_{\vec{P}}} dx \right]^\theta \left[\sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{E_3} \tilde{1}_{I_{\vec{P}}} dx \right]^{1-\theta} + 2^k \\ & \lesssim 2^k. \end{aligned}$$

Proof. By triangle inequality, it suffices to handle the sum

$$\begin{aligned} & \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ & + \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}: \omega_{Q_3} \supseteq \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ & = I + II. \end{aligned}$$

The Biest size estimate handles term II. For term I, use John-Nirenberg to observe

$$I \lesssim \sup_{\vec{P} \in T \cap \mathbb{P}_k} \frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right| \tilde{1}_{I_{\vec{P}}} dx.$$

Because $\vec{P} \in T \cap \mathbb{P}_k$, $I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^k \right\}^c \neq \emptyset$ and

$$\frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right| \tilde{1}_{I_{\vec{P}}} dx \lesssim 2^k.$$

This observation concludes the proof. □

We must therefore contend with exponential growth in the sizes of the trees in our tile collections \mathbb{P}_k . What makes this growth acceptable is the observation that for every $k \geq 0$

$$\bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \subset \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \gtrsim 2^k \right\}.$$

Therefore, $\left| \bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \right| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$ arises as a simple consequence of the $L^1 \rightarrow L^{1,\infty}$ bounds for the Hardy-Littlewood maximal function together with *BHT* estimates. Not surprisingly, it is the smallness of the support of the intervals in \mathbb{P}_k that will allow us to accept the largeness of the tree sizes. Recall the standard tile decomposition:

Lemma 10. *There is a decomposition $\mathbb{P} = \bigcup_{n_4 \geq 0} \mathbb{P}_{n_4,4}$ into disjoint subcollections with the property that if $\mathbb{I}_{n_4,4} := \{I_{\vec{P}} : \vec{P} \in \mathbb{P}_{n_4,4}\}$ then*

$$\begin{aligned} \text{Size}_1(f_4, \mathbb{P}_{n_4,4}) &\lesssim 2^{-n_4} \\ \sum_{I \in \mathbb{I}_{n_4,4}} |I| &\lesssim 2^{n_4} |E_4|. \end{aligned}$$

Proof. Initialize when $n_4 = 0$. Let $\mathbb{P}_{0,4} = \{\vec{P} \in \mathbb{P} : I_{\vec{P}} \subset \{M1_{E_4} \gtrsim 1\}\}$ and iteratively construct

$$\mathbb{P}_{n_4,4} := \left\{ \vec{P} \in \mathbb{P} \cap \left[\bigcup_{0 \leq m < n_4} \mathbb{P}_{m,4} \right]^c : I_{\vec{P}} \subset \{M1_{E_4} \geq 2^{-n_4}\} \right\}.$$

By John-Nirenberg, it is easy to check the desired properties. \square

The next step is to fix $k \geq 0$ and perform a size-energy stopping time decomposition in \mathbb{P}_k .

Lemma 11. *Fix $\tilde{d}, k, n_4 \geq 0$. Let \mathbb{I}_{k,n_4} be the collection of maximal (shifted) dyadic intervals in the collection*

$$\{I \subset \{M1_{E_4} \geq 2^{-n_4}\} : \exists \vec{P} \in \mathbb{P}_k \text{ s.t. } I = I_{\vec{P}}\}.$$

Then there exist two decompositions of \mathbb{P}_k , namely $\bigcup_{n_1 \geq 0} \tilde{\mathbb{P}}_{k,n_1,n_4,1}$ and $\bigcup_{\mathfrak{d} \geq -k} \tilde{\mathbb{P}}_{k,\mathfrak{d},2}^{\tilde{d}}$ such that $\text{Size}_1(f_1, \tilde{\mathbb{P}}_{k,n_1,n_4,1}) \lesssim 2^{-n_1}$ and $\text{Size}_2^{\tilde{d}}(f_2, f_3, \tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}_{n_1,1}^d$ and $\tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}}$ can each be written as a union of trees, i.e.

$$\begin{aligned} \tilde{\mathbb{P}}_{k,n_1,1} &= \bigcup_{T \in \mathcal{T}_{k,n_1,n_4,1}} \bigcup_{\vec{P} \in T} \vec{P} \\ \tilde{\mathbb{P}}_{k,\mathfrak{d},2}^{\tilde{d}} &= \bigcup_{T \in \mathcal{T}_{k,\mathfrak{d},2}^{\tilde{d}}} \bigcup_{\vec{P} \in T} \vec{P}, \end{aligned}$$

such that

$$\begin{aligned} \sum_{T \in \mathcal{T}_{k,n_1,n_4,1}^d} |I_T| &\lesssim 2^{2n_1} \sum_{T \in \mathcal{T}_{k,n_1,n_4,1,*}^d} \sum_{\vec{P} \in T} |\langle f_1, \Phi_{P_1,1} \rangle|^2 \\ \sum_{T \in \mathcal{T}_{k,\mathfrak{d},2}^{\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^d}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2, \end{aligned}$$

where $\mathcal{T}_{k,n_1,n_4,1,*} \subset \mathcal{T}_{k,n_1,n_4,1}$ and $\mathcal{T}_{k,\mathfrak{d},2,*}^{\tilde{d}} \subset \mathcal{T}_{k,\mathfrak{d},2}^{\tilde{d}}$, each tree in $\mathcal{T}_{k,n_1,1,*}$ is a 2-tree and each tree in $\mathcal{T}_{k,\mathfrak{d},2,*}^{\tilde{d}}$ is a 1-tree, and the collections $\mathcal{T}_{k,n_1,n_4,1,*}$, $\mathcal{T}_{k,\mathfrak{d},2,*}^{\tilde{d}}$ can each be written as the union of two strongly 2-disjoint subcollections. We denote this property by

$$\begin{aligned} \mathcal{T}_{k,n_1,1,*} &= \mathcal{T}_{k,n_1,1,*,+} \cup \mathcal{T}_{k,\mathfrak{d},2,*,+} \\ \mathcal{T}_{k,\mathfrak{d},2,*}^{\tilde{d}} &= \mathcal{T}_{k,\mathfrak{d},2,*,+}^{\tilde{d}} \cup \mathcal{T}_{k,\mathfrak{d},2,*,+}^{\tilde{d}}. \end{aligned}$$

Lastly, $\left[\bigcup_{T \in \mathcal{T}_{k,n_1,n_4,1,*}} I_T \right] \cup \left[\bigcup_{T \in \mathcal{T}_{k,n_1,n_4,1,*}} I_T \right] \subset \bigcup_{I \in \mathbb{I}_{k,n_4}} I$.

Proof. Apply the argument localized to each dyadic interval $I \in \mathbb{I}_k$ from Lemma 1 and use the John-Nirenberg inequality to impose the desired size restrictions. \square

11.1 Energy Savings

We shall now use $\left| \bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \right| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$ to improve our standard energy estimate. So, fix $\epsilon_0 > 0$, quite small perhaps. We first produce an additional energy decay factor of $2^{-k\epsilon_0}$. Start with

$$\sum_{T \in \mathbb{T}_k} |I_T| \lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathbb{T}_k} \sum_{\vec{P} \in T} \left| \left\langle \sum_{\omega_{Q_3} \subset \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2.$$

The trick here is to again majorize the *RHS* of the above display by

$$\begin{aligned} & 2 \cdot 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} \sum_{\vec{P} \in T} \left| \left\langle \sum_{\omega_{Q_3} \supseteq \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \\ & + 2 \cdot 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} \sum_{\vec{P} \in T} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2. \end{aligned}$$

Then using the localized Biest energy and standard BHT energy estimates,

$$\begin{aligned} I & \lesssim 2^{2\mathfrak{d}} \sum_{I \in \mathbb{I}_{k, n_4}} \|f_2 \tilde{I}_I\|_4^2 \|f_3 \tilde{I}_I\|_4^2 \\ II & \lesssim 2^{2\mathfrak{d}} \sum_{I \in \mathbb{I}_{k, n_4}} \|BHT(f_2, f_3) \tilde{I}_I\|_2^2. \end{aligned}$$

Since $\sum_{I \in \mathbb{I}_{k, n_4}} |I| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$,

$$\sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} |I_T| \lesssim 2^{2\mathfrak{d}} 2^{-k/2} |E_2|^{1/2} |E_3|^{1/2}.$$

As $\sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} |I_T| \lesssim_{\epsilon} 2^{2\mathfrak{d}} 2^{\tilde{d}} |E_2|^{2-\epsilon}$ and $\sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} |I_T| \lesssim 2^{\frac{2}{1-\epsilon}} |E_2|^{1/2} |E_3|^{1/2}$ (coming from the l^1 energy boost),

$$\sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} |I_T| \lesssim_{\epsilon, \tilde{\epsilon}} \min \left\{ 2^{2\mathfrak{d}} 2^{-k/2} |E_2|^{1/2} |E_3|^{1/2}, 2^{2\mathfrak{d}} 2^{\tilde{d}} |E_2|^{2-\epsilon}, 2^{\frac{2}{1-\epsilon}} |E_2|^{1/2} |E_3|^{1/2} \right\}$$

Similarly, we have $\sum_{T \in \mathcal{T}_{k, \mathfrak{d}, 2, *}} |I_T| \lesssim \min \{ 2^{2n_1} |E_1|, 2^{2n_1} 2^{n_4} |E_4| \}$. Indeed, the localized BHT energy yields

$$\begin{aligned} \sum_{T \in \mathcal{T}_{k, n_1, n_2, 1}} |I_T| & \lesssim 2^{2n_1} \sum_{I \in \mathbb{I}_{k, n_1}} \|f_1 \tilde{I}_I\|_2^2 \\ & \lesssim 2^{2n_1} \sum_{I \in \mathbb{I}_{k, n_1}} |I| \\ & \lesssim 2^{2n_1} 2^{n_4} |E_4|. \end{aligned}$$

Putting it all together gives

$$\begin{aligned} & \left| \Lambda_{\vec{P}, \vec{Q}}(f_1, f_2, f_3 1_{\Omega^c}, f_4) \right| \\ & \lesssim_{\epsilon, \tilde{\epsilon}} \sum_{\vec{d}, n_1, n_4, k \geq 0} \sum_{\mathfrak{d} \geq -k} 2^{-n_1} 2^{-n_4} 2^{-\mathfrak{d}} \min \left\{ 2^{2n_1} |E_1|, 2^{2n_1} 2^{n_4} |E_4|, 2^{2\mathfrak{d}} 2^{-k/2} |E_2|^{1/2}, 2^{2\mathfrak{d}} 2^{\tilde{d}} |E_2|^{2-\epsilon}, 2^{\frac{2}{1-\epsilon}} |E_3|^{1/2} \right\}. \end{aligned}$$

Take $\tilde{\epsilon} \simeq 0$. Using $0 \leq \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \leq 1$ to denote the weightings assigned to each term in the above minimum, we may deduce suitable weak type estimates in a neighborhood of $A_8 = (0, 1, -\frac{1}{2}, \frac{1}{2})$ by taking $\theta_1 \simeq 0, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $A_9 = (\frac{1}{2}, 1, -\frac{1}{2}, 0)$, take $\theta_1 \simeq \frac{1}{2}, \theta_2 \simeq 0, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. Some care has to be taken to ensure summability over $k \geq 0$. That this is possible is nonetheless straightforward, and so details are left to the reader. This concludes the proof of Theorem 5. \square

12 $C^{1,1,1-2}$ Estimates

Our goal in this section is to prove

Theorem 8. $C^{1,1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \times L^{p_4}(\mathbb{R}) \rightarrow L^{p'_4}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p'_4 < \infty$ and $(p_1, p_2, p_3, p_4) \in \text{Int}(\text{Conv}[\mathcal{B}]) \cap \text{Int}(\text{Conv}[\mathcal{B}'])$, where $\mathcal{B} = \{B_j\}_{j=1}^{16}$ is given by

$$\begin{aligned} B_1 &= \left(1, 1, \frac{1}{2}, \frac{1}{2}, -2\right), B_2 = \left(1, \frac{1}{2}, \frac{1}{2}, 1, -2\right), B_3 = \left(1, \frac{1}{2}, 1, \frac{1}{2}, -2\right) \\ B_4 &= \left(-2, 1, \frac{1}{2}, \frac{1}{2}, 1\right), B_5 = \left(-2, \frac{1}{2}, 1, \frac{1}{2}, 1\right), B_6 = \left(-2, \frac{1}{2}, \frac{1}{2}, 1, 1\right) \\ B_7 &= \left(0, -\frac{3}{2}, \frac{1}{2}, 1, 1\right), B_8 = \left(1, -\frac{3}{2}, \frac{1}{2}, 1, 0\right), B_9 = \left(0, -\frac{3}{2}, 1, \frac{1}{2}, 1\right), B_{10} = \left(1, -\frac{3}{2}, 1, \frac{1}{2}, 0\right) \\ B_{11} &= \left(0, \frac{1}{2}, -\frac{1}{2}, 1, 0\right), B_{12} = \left(\frac{1}{2}, 0, -\frac{1}{2}, 1, 0\right), B_{13} = \left(0, 0, -\frac{1}{2}, 1, \frac{1}{2}\right) \\ B_{14} &= \left(0, \frac{1}{2}, 1, -\frac{1}{2}, 0\right), B_{15} = \left(\frac{1}{2}, 0, 1, -\frac{1}{2}, 0\right), B_{16} = \left(0, 0, 1, -\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

Let \mathcal{B}' denote the collection $\{B'_j\}_{j=1}^{16}$, where each B'_j is obtained from the corresponding B_j by the permutation $1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 2, 4 \mapsto 3$. In particular, $B'_3 = (1, 1, \frac{1}{2}, \frac{1}{2}, -2)$. Moreover, $(1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3} - 2) \in \overline{\text{Conv}[\mathcal{B}]} \cap \overline{\text{Conv}[\mathcal{B}]'}$ and $C^{1,1,1,-2}$ maps into $L^r(\mathbb{R})$ for all $\frac{1}{3} < r \leq 1$.

12.1 Reduction to the Λ_3 Model

We proceed as in the proof of $C^{1,1,-2}$ estimates by decomposing $\left\{\xi_1 < \xi_2 < \xi_3 < -\frac{\xi_4}{2}\right\}$ into the following regions (viewed as subsets of $\left\{\xi_1 < \xi_2 < \xi_3 < -\frac{\xi_4}{2}\right\}$):

$$\begin{aligned}
\mathcal{R}_0 &= \left\{ |\xi_1 - \xi_2| \gg |\xi_2 - \xi_3| \simeq \left| \xi_3 + \frac{\xi_4}{2} \right| \right\} \\
\mathcal{R}_1 &= \left\{ |\xi_1 - \xi_2| \gg |\xi_2 - \xi_3| \gg \left| \xi_3 + \frac{\xi_4}{2} \right| \right\} \\
\mathcal{R}_2 &= \left\{ |\xi_1 - \xi_2| \gg \left| \xi_3 + \frac{\xi_4}{2} \right| \gg |\xi_2 - \xi_3| \right\} \\
\mathcal{R}_3 &= \left\{ |\xi_1 - \xi_2| \simeq |\xi_2 - \xi_3| \gg \left| \xi_3 + \frac{\xi_4}{2} \right| \right\} \\
\mathcal{R}_4 &= \left\{ |\xi_1 - \xi_2| \simeq \left| \xi_3 + \frac{\xi_4}{2} \right| \gg |\xi_2 - \xi_3| \right\} \\
\mathcal{R}_5 &= \left\{ |\xi_1 - \xi_2| \simeq |\xi_2 - \xi_3| \simeq \left| \xi_3 + \frac{\xi_4}{2} \right| \right\} \\
\mathcal{R}_6 &= \left\{ \left| \xi_3 + \frac{\xi_4}{2} \right| \gg |\xi_2 - \xi_3| \gg |\xi_1 - \xi_2| \right\} \\
\mathcal{R}_7 &= \left\{ \left| \xi_2 + \frac{\xi_4}{2} \right| \gg |\xi_1 - \xi_2| \gg |\xi_2 - \xi_3| \right\} \\
\mathcal{R}_8 &= \left\{ \left| \xi_3 + \frac{\xi_4}{2} \right| \simeq |\xi_2 - \xi_3| \gg |\xi_1 - \xi_2| \right\} \\
\mathcal{R}_9 &= \left\{ \left| \xi_3 + \frac{\xi_4}{2} \right| \gg |\xi_2 - \xi_3| \simeq |\xi_1 - \xi_2| \right\} \\
\mathcal{R}_{10} &= \{ |\xi_2 - \xi_3| \gg |\xi_1 - \xi_2| \} \cap \left\{ |\xi_2 - \xi_3| \gg \left| \xi_3 + \frac{\xi_4}{2} \right| \right\}.
\end{aligned}$$

One expects that the most problematic regions are \mathcal{R}_1 and \mathcal{R}_2 and that, by symmetry, it should suffice to handle generalized restricted type estimates for the discretized model of generic symbols adapted to \mathcal{R}_1 , say. In light of previous work with both $C^{-1,1,-1}$ and $C^{1,1,-2}$, it should come as no surprise that the most natural symbol $\tilde{1}_{\mathcal{R}_1}$ localized to \mathcal{R}_1 and identically equal to 1 on a subregion of the same shape cone can be written as a sum of expressions all of the generic form

$$\sum_{\vec{P} \in \mathcal{P}} \sum_{\vec{Q} \in \mathcal{Q}: |\vec{Q}| < |\vec{P}|} \sum_{\vec{R} \in \mathcal{R}: \widetilde{\omega_{R_1}} \supset \supset \omega_{P_2}} \hat{\eta}_{R_1,1}(\xi_1) \hat{\eta}_{R_3,0}(\xi_2 + \xi_3 + \xi_4) \hat{\eta}_{P_2,2}(\xi_2) \hat{\eta}_{P_3,7}(\xi_3 + \xi_4),$$

where each $\vec{R} = (R_1, R_3)$ is a frequency square intersecting the line $\{\xi_2 = 0\}$, each $\vec{P} = (P_1, P_2)$ is a frequency square adapted to the line $\{\xi_1 + \xi_2 = 0\}$, and each $\vec{Q} = (Q_1, Q_2)$ is a frequency square adapted to $\{\xi_1 = -\xi_2/2\}$. As usual, we denote $\tilde{I} := I + C|I|$ for some large $C \gg 1$. We may now dualize by introducing another function f_5 and then complete the resulting integral as follows:

$$\begin{aligned}
&\Lambda(f_1, f_2, f_3, f_4, f_5) \\
&:= \int_{\mathbb{R}} \sum_{\vec{P} \in \mathcal{P}} \sum_{\vec{Q} \in \mathcal{Q}: |\vec{Q}| < |\vec{P}|} \sum_{\vec{R} \in \mathcal{R}: \widetilde{\omega_{R_1}} \supset \supset \omega_{P_2}} f_1 * \eta_{R_1,1} [f_2 * \eta_{P_2,2} [f_3 * \eta_{Q_1,3} f_4 * \eta_{Q_2,4}] * \eta_{P_3,7}] * \eta_{-R_3,0} f_5 dx \\
&= \int_{\mathbb{R}} \sum_{\vec{P} \in \mathcal{P}} \sum_{\vec{Q} \in \mathcal{Q}: |\vec{Q}| < |\vec{P}|} \sum_{\vec{R} \in \mathcal{R}: \widetilde{\omega_{R_1}} \supset \supset \omega_{P_2}} [f_1 * \eta_{R_1,1} f_5 * \eta_{R_2,5}] * \tilde{\eta}_{R_3,0} f_2 * \eta_{P_2,2} [f_3 * \eta_{Q_1,3} f_4 * \eta_{Q_2,4}] * \eta_{P_3,7} dx,
\end{aligned}$$

which can then be discretized in the standard way to yield a sum over rapidly decaying terms of averages of generic forms of type Λ_3 given by

$$\sum_{\vec{P} \in \mathcal{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R} \in \mathcal{R}: \widetilde{\omega_{R_1}} \supset \supset \omega_{P_2}} \frac{1}{|I_{\vec{R}}|^{1/2}} \langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle \Phi_{R_3,0}^{n-l} \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle.$$

While \mathbb{Q} is rank-1 collection of tri-tiles, both \mathbb{R} and \mathbb{P} are not. We now show generalized restricted type estimates for type Λ_3 models. Similar to the arguments presented in proof of Proposition 3, we may reduce the proof of Theorem 8 to the proof of its discretized version:

Theorem 9. *Let $\sigma, \sigma', \tilde{\sigma} \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be shifts, and let $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ be finite collections of tri-tiles with shifts $\sigma, \sigma', \tilde{\sigma}$ respectively so that \mathbb{Q} is rank-1. Define the form $\Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}$ by*

$$\Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}(f_1, f_2, f_3, f_4, f_5) = \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R} \in \mathbb{R}: \widetilde{\omega_{R_1}} \supset \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle}{|I_{\vec{R}}|^{1/2}} \Phi_{R_3,0}^{n-l}, \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle,$$

where the \mathbb{P} -sum is over all tri-tiles of the form $\vec{P} = (P_1, P_2, P_3)$, $\Phi_{P_1,1}$ is a wave packet on $I_{\vec{P}} \times \omega_{|\vec{P}|}^{lac} := I_{\vec{P}} \times [c_1 |I_{\vec{P}}|^{-1}, c_2 |I_{\vec{P}}|^{-1}]$ for some absolute constants $c_1 < c_2$, $\Phi_{P_2,2}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_2}$, $\Phi_{P_3,7}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_3}$, and

$$BHT_{\omega_{P_3}}^{\alpha}(f_2, f_3)(x) := \sum_{\vec{Q} \in \mathbb{Q}: |\omega_{Q_1}| < |\omega_{P_1}|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,3}^{\alpha} \rangle \Phi_{Q_3,5}^{\alpha}(x),$$

where for each $\alpha \in [0, 1]$, the \mathbb{Q} -sum is over all tri-tiles of the form $\vec{Q} = (Q_1, Q_2, Q_3)$, $\Phi_{Q_1,2}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_1}$, $\Phi_{Q_2,3}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_2}$, $\Phi_{Q_3,5}^{\alpha}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_3}$. Then $\Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}$ is restricted type $\vec{\alpha}$ for all admissible tuples in $\vec{\alpha} \in \text{Int}[\text{Conv}[\mathcal{B}]]$, uniformly in the parameters

$$\sigma, \sigma', \mathbb{P}, \mathbb{Q}, \{\Phi_{P_i,j(i)}\}, \{\Phi_{Q_i,j(i)}^{\alpha}\}.$$

For completeness, we record

Proposition 5. *To prove Theorem 8, it suffices to prove Theorem 9.*

Proof. The numerous details required for a full demonstration are a bit tedious to verify and very similar to the proof of Proposition 3. \square

Note as before that if $\vec{\alpha} \in \text{Int}(\text{Conv}[\mathcal{B}])$ has a bad index j , the restricted type estimate will *not* necessarily be uniform in the sense that the major subset E_j' cannot be chosen independently of the parameters just mentioned. Proposition 5 ensures that taking the expectational set independent of such parameters is not an obstacle in reducing Theorem 8 to Theorem 9. We now prove Theorem 9.

Proof. **13 Generalized Restricted Weak Estimates near B_1, B_2, B_3**

13.1 Tile Decomposition

Fix dyadic shifts $\sigma, \sigma', \tilde{\sigma}$ and corresponding tri-tile collections \mathbb{P}, \mathbb{Q} , and \mathbb{R} once and for all. By assumption, \mathbb{Q} is rank-1. Moreover, for convenience, we shall subsequently use f_j to denote f_j' for $j = 1, 2, 3, 4$ in Theorem 9 and assume by rescaling that $|E_5| = 1$ and the collections $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are sparse. Next, set

$$\tilde{\Omega} = \{M1_{E_3} \gtrsim |E_3|\} \cup \{M1_{E_4} \gtrsim |E_4|\}.$$

For each $\tilde{d} \geq 0$ set $\mathbb{Q}^{\tilde{d}} := \left\{ \vec{Q} \in \mathbb{Q} : 1 + \frac{\text{dist}(I_{\vec{Q}}, \tilde{\Omega}^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}} \right\}$ and define for $0 < \theta < 1$

$$\begin{aligned} \Omega_1^{\tilde{d}} &= \left\{ M \left[\int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_3, f_4) d\alpha \right] \gtrsim_{\theta} 2^{\tilde{d}} |E_3|^{1/2} |E_4|^{1/2} \right\} \\ \Omega_2^{\tilde{d}} &= \left\{ M \left(\left[\int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_3, \Phi_{Q_1,3}^{\alpha} \rangle \langle f_4, \Phi_{Q_2,4}^{\alpha} \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \right]^2 \right) \gtrsim 2^{2\tilde{d}} |E_3| |E_4| \right\}. \end{aligned}$$

Lastly, construct

$$\Omega = \bigcup_{\tilde{d} \geq 0} \Omega_1^{\tilde{d}} \bigcup_{\tilde{d} \gtrsim 1} \Omega_2^{\tilde{d}} \bigcup \{M1_{E_1} \gtrsim |E_1|\} \bigcup \{M1_{E_2} \gtrsim |E_2|\} \bigcup \tilde{\Omega}.$$

Then for large enough implicit constants depending on ϵ , $|\Omega(\epsilon)| \leq 1/2$ and $\tilde{E}_5 := E_5 \cap \Omega(\epsilon)^c$ is a major subset of E_5 since $|E_5| = 1$. Now let $\mathbb{P}^d := \left\{ \vec{P} \in \mathbb{P} : 1 + \frac{\text{dist}(I_{\vec{P}}, \Omega(\epsilon)^c)}{|I_{\vec{P}}|} \simeq 2^d \right\}$. Assuming $|f_1| \leq 1_{E_1}, |f_2| \leq 1_{E_2}, |f_3| \leq 1_{E_3}, |f_4| \leq 1_{E_4}, |f_5| \leq 1_{E_5 \cap \Omega^c}$, recall that our task in this section is to obtain the estimate $|\Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}(f_1, f_2, f_3, f_4, f_5)| \lesssim |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in a small neighborhood near an extremal point $\vec{\beta} \in \{B_1, B_2, B_3\}$.

$$\begin{aligned} & \Lambda_{\mathbb{P}, \mathbb{Q}, \mathbb{R}}(f_1, f_2, f_3, f_4, f_5) \\ &= \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R} \in \mathbb{R}(T)} \frac{1}{|I_{\vec{R}}|^{1/2}} \langle f_1, \Phi_{R_{1,1}} \rangle \langle f_5, \Phi_{R_{2,5}} \rangle \Phi_{R_{3,0}}^{n-l}, \Phi_{P_{1,6}} \right\rangle \langle f_2, \Phi_{P_{2,2}} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_{3,7}} \right\rangle, \end{aligned}$$

where \mathbb{P}, \mathbb{Q} , and \mathbb{R} are three tri-tile collections, with the additional assumption that \mathbb{Q} is rank-1. For any subcollection of tri-tiles $\tilde{\mathbb{P}} \subset \mathbb{P}$, let

$$\begin{aligned} \text{Size}_2(f_2, \tilde{\mathbb{P}}) &:= \sup_{T \subset \tilde{\mathbb{P}}} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} |\langle f_2, \Phi_{P_{2,2}} \rangle|^2 \right)^{1/2} \\ \text{Size}_7^{\tilde{d}}(f_3, f_4, \tilde{\mathbb{P}}) &:= \sup_{T \subset \tilde{\mathbb{P}}} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_3, f_4) d\alpha, \Phi_{P_{3,7}} \right\rangle \right|^2 \right)^{1/2}, \end{aligned}$$

where the supremum arising in the definition of the 1-Size is over all 3-trees and the supremum arising in the definition of the 7-Size is over all 2-trees. As before, both sizes generate decompositions of \mathbb{P}^d for each $\tilde{d} \geq 0$, namely $\bigcup_{n_2 \geq N_2(d)} \tilde{\mathbb{P}}_{n_2,2}^d$ and $\bigcup_{\mathfrak{d} \geq N_3(d, \tilde{d})} \mathbb{P}_{\mathfrak{d},3}^{d, \tilde{d}}$ such that $\text{Size}_2(f_2, \tilde{\mathbb{P}}_{n_2,2}^d) \lesssim 2^{-n_2}$ and $\text{Size}_7^{\tilde{d}}(f_3, f_4, \mathbb{P}_{\mathfrak{d},3}^{d, \tilde{d}}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}_{n_2,2}^d$ and $\mathbb{P}_{\mathfrak{d},3}^{d, \tilde{d}}$ can each be written as a union of trees, i.e.

$$\begin{aligned} \tilde{\mathbb{P}}_{n_2,2}^d &= \bigcup_{T \in \mathcal{T}_{n_2,2}^d} \bigcup_{\vec{P} \in T} \vec{P} \\ \mathbb{P}_{\mathfrak{d},3}^{d, \tilde{d}} &= \bigcup_{T \in \mathcal{T}_{\mathfrak{d},3}^{d, \tilde{d}}} \bigcup_{\vec{P} \in T} \vec{P}, \end{aligned}$$

such that

$$\begin{aligned} \sum_{T \in \mathcal{T}_{n_2,2}^d} |I_T| &\lesssim 2^{2n_2} \sum_{T \in \mathcal{T}_{n_2,2,*}^d} \sum_{\vec{P} \in T} |\langle f_1, \Phi_{P_{2,2}} \rangle|^2 \\ \sum_{T \in \mathcal{T}_{\mathfrak{d},3}^{d, \tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},3,*}^{d, \tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_{2,0}} \right\rangle \right|^2, \end{aligned}$$

where $\mathcal{T}_{n_2,2,*}^d \subset \mathcal{T}_{n_2,2}^d$ and $\mathcal{T}_{\mathfrak{d},3,*}^{d, \tilde{d}} \subset \mathcal{T}_{\mathfrak{d},3}^{d, \tilde{d}}$, each tree in $\mathcal{T}_{n_2,2,*}^d$ is a 3-tree and each tree in $\mathcal{T}_{\mathfrak{d},3,*}^{d, \tilde{d}}$ is a 2-tree, and the collections $\mathcal{T}_{n_2,2,*}^d, \mathcal{T}_{\mathfrak{d},3,*}^{d, \tilde{d}}$ can each be written as the union of a strongly 2-disjoint and 3-disjoint subcollections respectively. We denote this last property by

$$\begin{aligned} \mathcal{T}_{n_1,1,*}^d &= \mathcal{T}_{n_1,1,*,+}^d \bigcup \mathcal{T}_{n_1,1,*, -}^d \\ \mathcal{T}_{\mathfrak{d},2,*}^{d, \tilde{d}} &= \mathcal{T}_{\mathfrak{d},2,*,+}^{d, \tilde{d}} \bigcup \mathcal{T}_{\mathfrak{d},2,*, -}^{d, \tilde{d}}. \end{aligned}$$

Similar to before, construct $\mathbb{P}_{n_2,\mathfrak{d}}^{d, \tilde{d}} = \tilde{P}_{n_2,2}^{d, \tilde{d}} \cap \tilde{P}_{\mathfrak{d},3}^{d, \tilde{d}}$

13.2 Tree Estimate

First, we should recall from [12] the following statement:

Lemma 12. *Let $\tilde{\mathbb{P}} \subset \mathbb{P}$ be any sub collection of tri-tiles. Then, for any $0 < \theta < 1$ and 1-tree $T \subset \tilde{\mathbb{P}}$,*

$$\left[\frac{1}{|I_T|} \sum_{\tilde{P} \in T} \left| \left\langle \int_0^1 \sum_{\tilde{Q} \in \mathbb{Q}^{\tilde{d}}: \omega_{Q_3} \supset \omega_{P_3}} \frac{1}{|I_{\tilde{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^\alpha \rangle \langle f_3, \Phi_{Q_2,3}^\alpha \rangle \Phi_{Q_3,5}^\alpha d\alpha, \Phi_{P_3,7} \rangle \right| \right]^2 \right]^{1/2} \\ \lesssim_\theta \left[\sup_{\tilde{P} \in \tilde{\mathbb{P}}} \frac{1}{|I_{\tilde{P}}|} \int 1_{E_2} \tilde{1}_{I_{\tilde{P}}} dx \right]^\theta \left[\sup_{\tilde{P} \in \tilde{\mathbb{P}}} \frac{1}{|I_{\tilde{P}}|} \int 1_{E_3} \tilde{1}_{I_{\tilde{P}}} dx \right]^{1-\theta}.$$

Proof. See [12]. □

Now, letting $T \subset \mathbb{P}_{n_2, \mathfrak{d}}^{d, \tilde{d}}$ be a 3-tree, we use the proceeding lemma to conclude

$$\left| \sum_{\tilde{P} \in \mathbb{P}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \left\langle \sum_{\tilde{R} \in \mathbb{R}: \omega_{R_1} \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle \Phi_{R_3,0}^{n-l}, \Phi_{P_1,6} \rangle \langle f_2, \Phi_{P_2,2} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle \right| \\ \leq \frac{\left(\sum_{\tilde{P} \in T} \left| \left\langle \sum_{\tilde{R} \in \mathbb{R}: \omega_{R_1} \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle \Phi_{R_3,0}^{n-l}, \Phi_{P_1,6} \rangle \right| \right)^2 \right)^{1/2}}{|I_T|^{1/2}} \cdot \frac{(\sum_{\tilde{P} \in T} |\langle f_2, \Phi_{P_2,2} \rangle|^2)^{1/2}}{|I_T|^{1/2}} \\ \times \sup_{\tilde{P} \in T} \left[\frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_3, f_4) d\alpha, \tilde{\Phi}_{P_3,7}^\infty \right\rangle \right|}{|I_{\tilde{Q}}|} \right] |I_T| \\ \lesssim_\theta 2^{-\tilde{N}d(1-\theta)} |E_1|^\theta 2^{-n_2} 2^{-\mathfrak{d}} |I_T|.$$

If $T \subset \mathbb{P}_{n_2, \mathfrak{d}}^{d, \tilde{d}}$ be a 2-tree, then

$$\left| \sum_{\tilde{P} \in \mathbb{P}} \frac{1}{|I_{\tilde{P}}|^{1/2}} \left\langle \sum_{\tilde{R} \in \mathbb{R}: \omega_{R_1} \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle \Phi_{R_3,0}^{n-l}, \Phi_{P_1,6} \rangle \langle f_2, \Phi_{P_2,2} \rangle \left\langle \int_0^1 BHT_{\omega_{P_3}}^{\alpha, \mathbb{Q}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle \right| \\ \lesssim \left[\sup_{\tilde{P} \in T} \frac{\left| \left\langle \sum_{\tilde{R} \in \mathbb{R}: \omega_{R_1} \supset \omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1} \rangle \langle f_5, \Phi_{R_2,5} \rangle \Phi_{R_3,0}^{n-l}, \Phi_{P_1,6} \rangle \right|}{|I_{\tilde{P}}|^{1/2}} \right] \left(\sum_{\tilde{P} \in T} \frac{|\langle f_4, \Phi_{\tilde{P},4}^{lac} \rangle|^2}{|I_T|} \right)^{1/2} \\ \times \left(\sum_{\tilde{P} \in T} \frac{\left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle \right|^2}{|I_T|} \right)^{1/2} |I_T| \\ \lesssim_\theta 2^{-\tilde{N}d(1-\theta)} |E_1|^\theta 2^{-n_2} 2^{-\mathfrak{d}} |I_T|.$$

13.3 Size Restrictions

Lemma 13. *Fix $d, \tilde{d}, n_2, \mathfrak{d}$ such that $\mathbb{P}_{n_2, \mathfrak{d}}^{d, \tilde{d}}$ is nonempty. Then*

$$2^{-n_2} \lesssim 2^d |E_2| \\ 2^{-\mathfrak{d}} \lesssim 2^{-\tilde{N}(\tilde{d}-d)} |E_3|^{1/2} |E_4|^{1/2}.$$

Proof. The proof is identical to the previous size restriction argument and therefore omitted. □

13.4 Synthesis

The l^2 and l^1 energy estimates for $\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)$ are essentially identical to the $\Lambda_{\mathbb{P},\mathbb{Q}}(f_1, f_2, f_3, f_4)$ case, it suffices to assemble all the pieces. Using Theorem 6 as a guide, the reader may check that for all $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$

$$\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)| \\
& \lesssim_{\theta, \tilde{\epsilon}} \sum_{d, \tilde{d} \geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d, \tilde{d})} 2^{-\tilde{N}d(1-\theta)} 2^{d\theta} |E_1|^\theta 2^{-n_2} 2^{-\mathfrak{d}} \min \left\{ 2^{2n_2} |E_2|, 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_3|^{1+\tilde{\theta}} |E_4|^{2-\tilde{\theta}}, 2^{\frac{2}{1-\tilde{\epsilon}}} |E_3|^{1/2} |E_4|^{1/2} \right\} \\
& \lesssim \sum_{d, \tilde{d} \geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d, \tilde{d})} 2^{-\tilde{N}d(1-\theta)} 2^{d\theta} |E_1|^\theta 2^{-n_2} 2^{-\mathfrak{d}} 2^{2n_2\theta_2} 2^{2\mathfrak{d}\theta_2} 2^{2\tilde{d}\theta_2} |E_3|^{(1+\tilde{\theta})\theta_2} |E_4|^{(2-\tilde{\theta})\theta_2} 2^{\frac{2\theta_3}{1-\tilde{\epsilon}}} |E_3|^{\theta_3/2} |E_4|^{\theta_3/2} \\
& \leq \sum_{d, \tilde{d} \geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d, \tilde{d})} 2^{-\tilde{N}d(1-\theta)/2} |E_1|^\theta 2^{-n_2(1-2\theta_1)} 2^{-\mathfrak{d}[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}]} |E_3|^{(1+\tilde{\theta})\theta_2+\frac{\theta_3}{2}} |E_4|^{(2-\tilde{\theta})\theta_2+\frac{\theta_3}{2}}.
\end{aligned}$$

Take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $B_1 = (1, 1, \frac{1}{2}, \frac{1}{2}, -1)$, set $\theta \simeq 1, \tilde{\theta} = \frac{1}{2}, \theta_1 \simeq 0, \theta_2 \simeq 0, \theta_3 \simeq 1$. For $B_2 = (1, \frac{1}{2}, \frac{1}{2}, 1, -1)$, set $\theta \simeq 1, \tilde{\theta} \simeq 0, \theta_1 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$. Lastly, for $B_3 = (1, \frac{1}{2}, 1, \frac{1}{2} - 1)$, set $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq \frac{1}{2}, \theta_1 \simeq 0$.

14 Gen. Restricted Weak Estimates near $B_4, B_5, B_6, B_7, B_8, B_9, B_{10}$

The model $\Lambda_{\mathbb{P},\mathbb{R},\mathbb{Q}}$ is symmetric in positions 1 and 5, and so estimate near B_1, B_2, B_3 ensures estimates near B_4, B_5, B_6 . Generalized restricted type estimates near B_7, B_8, B_9, B_{10} follow from the following observations:

$$\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)| \\
& \lesssim_{\theta, \tilde{\epsilon}} \sum_{d, \tilde{d} \geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d, \tilde{d})} 2^d |E_1|^{1-\theta} |E_5|^\theta 2^{-n_2(1-2\theta_1)} 2^{-\mathfrak{d}[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}]} |E_3|^{(1+\tilde{\theta})\theta_2+\frac{\theta_3}{2}} |E_4|^{(2-\tilde{\theta})\theta_2+\frac{\theta_3}{2}}
\end{aligned}$$

Again take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $B_7 = (0, -\frac{3}{2}, \frac{1}{2}, 1, 1)$, set $\theta \simeq 1, \tilde{\theta} \simeq 0, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_8 = (1, -\frac{3}{2}, \frac{1}{2}, 1, 0)$, set $\theta \simeq 0, \tilde{\theta} \simeq 0, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_9 = (0, -\frac{3}{2}, 1, \frac{1}{2}, 1)$, set $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_{10} = (1, -\frac{3}{2}, 1, \frac{1}{2}, 0)$, set $\theta \simeq 0, \tilde{\theta} \simeq 1, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$.

15 Gen. Restricted Weak Estimates near $B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}$

By modifying the adjoint tile decomposition for $\Lambda_{\mathbb{P},\mathbb{Q}}$, it is not difficult to observe

$$\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3 1_{\Omega^c}, f_4)| \\
& \lesssim_{\theta, \tilde{\theta}, \tilde{\epsilon}} \sum_{\tilde{d}, n_1, n_4, k \geq 0} \sum_{\mathfrak{d} \geq -k} 2^{-n_1(1-\theta)} 2^{-n_5\theta} 2^{-n_4} 2^{-\mathfrak{d}} \\
& \times \min \left\{ 2^{2n_2} |E_2|, 2^{2n_2} 2^{n_1(1-\theta)} 2^{n_5\theta} |E_1|^{1-\theta} |E_5|^\theta, 2^{2\mathfrak{d}} 2^{-k/2} |E_4|^{1/2}, 2^{2\mathfrak{d}} 2^{\tilde{d}} |E_4|^{2-\tilde{\theta}}, 2^{\frac{2}{1-\tilde{\epsilon}}} |E_4|^{1/2} \right\}.
\end{aligned}$$

Take $\tilde{\epsilon} \simeq 0$. Using $0 \leq \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \leq 1$ to denote the weightings assigned to each term in the above minimum, we may deduce suitable weak type estimates in a neighborhood of $B_{11} = (0, \frac{1}{2}, -\frac{1}{2}, 1, 0)$ by taking $\theta = 1/2, \tilde{\theta} \simeq 0, \theta_1 \simeq 0, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $B_{12} = (\frac{1}{2}, 0, -\frac{1}{2}, 1, 0)$, take $\theta \simeq 0, \tilde{\theta} \simeq 0, \theta_1 \simeq 0, \theta_2 \simeq 1/2, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $B_{13} = (0, 0, -\frac{1}{2}, 1, \frac{1}{2})$, take $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_1 \simeq 0, \theta_2 \simeq 1/2, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. Some care has to be taken to ensure summability over $k \geq 0$. That this is possible is again straightforward, and so details are left to the reader. Generalized restricted type estimates near $B_{14} = (0, \frac{1}{2}, 1, -\frac{1}{2}, 0), B_{15} = (\frac{1}{2}, 0, 1, -\frac{1}{2}, 0), B_{16} = (0, 0, 1, -\frac{1}{2}, \frac{1}{2})$ are obtained by symmetry with B_{11}, B_{12}, B_{13} . This concludes the proof of Theorem 8. \square

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